CHAPTER TWO

4.8 Indefinite Integrals

DEFINITION Indefinite Integral, Integrand
The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

1. $\int dx = x + c$
2. $\int kf(x)dx = k\int f(x)dx$
3. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
4. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$ if n≠-1
5. $\int \left(f(x)\right)^n f(x)' dx = \dfrac{\left(f(x)^{n+1}\right)}{n+1} + c$ if n≠-1

Using this notation, we restate the solutions of Example 1, as follows:

$$\int 2x\, dx = x^2 + C,$$

$$\int \cos x\, dx = \sin x + C,$$

$$\int (2x + \cos x)\, dx = x^2 + \sin x + C.$$

EXAMPLE 7 Indefinite Integration Done Term-by-Term and Rewriting the Constant of Integration

Evaluate

$$\int (x^2 - 2x + 5)\,dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\,dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}
\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\
&= \int x^2\,dx - 2\int x\,dx + 5\int 1\,dx \\
&= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3) \\
&= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.
\end{aligned}$$

This formula is more complicated than it needs to be. If we combine C_1, $-2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

$$\begin{aligned}
\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\
&= \frac{x^3}{3} - x^2 + 5x + C.
\end{aligned}$$

Example: Evaluate

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{3/2}}{3/2} + c$$

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

1- $\int \cos u\,du = \sin u + c$

2- $\int \sin u\,du = -\cos u + c$

3- $\int \sec^2 u\,du = \tan u + c$

4- $\int \csc^2 u\,du = -\cot u + c$

5- $\int \sec u \tan u\,du = \sec u + c$

6- $\int \csc u \cot u\,du = -\csc u + c$

Example: Evaluate

$\int \tan x \sec^2 x\,dx$

Solution

$\frac{\tan^2 x}{2} + c$

EXERCISES 4.8

Finding Indefinite Integrals In Exercises 17–54, find the most general indefinite integral. Check your answers by differentiation.

17. $\int (x + 1)\,dx$

18. $\int (5 - 6x)\,dx$

19. $\int \left(3t^2 + \frac{t}{2}\right) dt$

20. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$

21. $\int (2x^3 - 5x + 7)\,dx$

22. $\int (1 - x^2 - 3x^5)\,dx$

23. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$

24. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

25. $\int x^{-1/3}\,dx$

26. $\int x^{-5/4}\,dx$

27. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$

28. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

29. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$

30. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$

31. $\int 2x(1 - x^{-3})\,dx$

32. $\int x^{-3}(x + 1)\,dx$

33. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\,dt$

34. $\int \frac{4 + \sqrt{t}}{t^3}\,dt$

35. $\int (-2\cos t)\,dt$

36. $\int (-5\sin t)\,dt$

37. $\int 7\sin\frac{\theta}{3}\,d\theta$

38. $\int 3\cos 5\theta\,d\theta$

39. $\int (-3\csc^2 x)\,dx$

40. $\int \left(-\frac{\sec^2 x}{3}\right)dx$

41. $\int \frac{\csc\theta\cot\theta}{2}\,d\theta$

42. $\int \frac{2}{5}\sec\theta\tan\theta\,d\theta$

43. $\int (4\sec x\tan x - 2\sec^2 x)\,dx$

44. $\int \frac{1}{2}(\csc^2 x - \csc x\cot x)\,dx$

45. $\int (\sin 2x - \csc^2 x)\,dx$

46. $\int (2\cos 2x - 3\sin 3x)\,dx$

47. $\int \frac{1 + \cos 4t}{2}\,dt$

48. $\int \frac{1 - \cos 6t}{2}\,dt$

49. $\int (1 + \tan^2\theta)\,d\theta$

(*Hint:* $1 + \tan^2\theta = \sec^2\theta$)

50. $\int (2 + \tan^2\theta)\,d\theta$

51. $\int \cot^2 x\,dx$

(*Hint:* $1 + \cot^2 x = \csc^2 x$)

52. $\int (1 - \cot^2 x)\,dx$

53. $\int \cos\theta\,(\tan\theta + \sec\theta)\,d\theta$

54. $\int \frac{\csc\theta}{\csc\theta - \sin\theta}\,d\theta$

Solution

17. $\int (x+1)\,dx = \frac{x^2}{2} + x + C$

18. $\int (5-6x)\,dx = 5x - 3x^2 + C$

19. $\int \left(3t^2 + \frac{t}{2}\right) dt = t^3 + \frac{t^2}{4} + C$

20. $\int \left(\frac{t^2}{2} + 4t^3\right) dt = \frac{t^3}{6} + t^4 + C$

25. $\int x^{-1/3}\,dx = \frac{x^{2/3}}{\frac{2}{3}} + C = \frac{3}{2}x^{2/3} + C$

26. $\int x^{-5/4}\,dx = \frac{x^{-1/4}}{-\frac{1}{4}} + C = \frac{-4}{\sqrt[4]{x}} + C$

28. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx = \int \left(\frac{1}{2}x^{1/2} + 2x^{-1/2}\right) dx = \frac{1}{2}\left(\frac{x^{3/2}}{\frac{3}{2}}\right) + 2\left(\frac{x^{1/2}}{\frac{1}{2}}\right) + C = \frac{1}{3}x^{3/2} + 4x^{1/2} + C$

33. $\int \frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt = \int \left(\frac{t^{3/2}}{t^2} + \frac{t^{1/2}}{t^2}\right) dt = \int \left(t^{-1/2} + t^{-3/2}\right) dt = \frac{t^{1/2}}{\frac{1}{2}} + \left(\frac{t^{-1/2}}{-\frac{1}{2}}\right) + C = 2\sqrt{t} - \frac{2}{\sqrt{t}} + C$

35. $\int -2\cos t\,dt = -2\sin t + C$

36. $\int -5\sin t\,dt = 5\cos t + C$

37. $\int 7\sin\frac{\theta}{3}\,d\theta = -21\cos\frac{\theta}{3} + C$

38. $\int 3\cos 5\theta\,d\theta = \frac{3}{5}\sin 5\theta + C$

43. $\int (4\sec x\tan x - 2\sec^2 x)\,dx = 4\sec x - 2\tan x + C$

47. $\int \frac{1+\cos 4t}{2}\,dt = \int \left(\frac{1}{2} + \frac{1}{2}\cos 4t\right) dt = \frac{1}{2}t + \frac{1}{2}\left(\frac{\sin 4t}{4}\right) + C = \frac{t}{2} + \frac{\sin 4t}{8} + C$

Integrations of the form $\int f(ax+b)\,dx$

For the integral $\int f(ax+b)\,dx$, make the substitution $u = ax + b$.

Example 7.5 Find $\int \sin(3x+2)\,dx$.

Solution Substitute $u = 3x + 2$. Then $du/dx = 3 \Rightarrow du = 3\,dx \Rightarrow dx = du/3$. Then the integral becomes

$$\int \sin(u)\frac{du}{3} = -\frac{\cos(u)}{3} + C$$

Re-substitute $u = 3x + 2$ to give

$$\int \sin(3x+2)\,dx = -\frac{\cos(3x+2)}{3} + C.$$

Check:

$$\frac{d}{dx}\left(-\frac{\cos(3x+2)}{3} + C\right) = \frac{\sin(3x+2)}{3}\frac{d}{dx}(3x+2)$$
$$= \frac{3\sin(3x+2)}{3} = \sin(3x+2).$$

Example 7.6 Integrate

$$\frac{1}{\sqrt{1-(3-x)^2}}$$

with respect to x.

Solution Notice that this is very similar to the expression which integrates to $\sin^{-1}(x)$ or $\cos^{-1}(x)$. We substitute for the expression in the bracket $u = 3 - x$ giving $du/dx = -1 \Rightarrow dx = -du$. The integral

becomes

$$\int \frac{1}{\sqrt{1-(u)^2}}(-du) = \int \frac{(-du)}{\sqrt{1-(u)^2}}$$

From Table 7.1, this integrates to give

$$\cos^{-1}(u) + C$$

Re-substituting $u = 3 - x$ gives

$$\int \frac{1}{\sqrt{1-(3-x)^2}}\,dx = \cos^{-1}(3-x) + C.$$

Check:

$$\begin{aligned}\frac{d}{dx}(\cos^{-1}(3-x)+C) &= -\frac{1}{\sqrt{1-(3-x)^2}}\frac{d}{dx}(3-x)\\ &= \frac{1}{\sqrt{1-(3-x)^2}}.\end{aligned}$$

Integrals of the form $\int f(u)(du/dx)\,dx$

Example 7.7 Find $\int x\sin(x^2)\,dx$.

Solution Substitute $u = x^2 \Rightarrow du/dx = 2x \Rightarrow du = 2x\,dx \Rightarrow dx = du/2x$ to give

$$\begin{aligned}\int x\sin(x^2)\,dx = \int x\sin(u)\frac{du}{2x} &= \int \frac{1}{2}\sin(u)\,du\\ &= -\frac{1}{2}\cos(u) + C.\end{aligned}$$

As $u = x^2$, we have

$$\int x\sin(x^2)\,dx = -\frac{1}{2}\cos(x^2) + C.$$

Check:

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{2}\cos(x^2)+C\right) &= \frac{1}{2}\sin(x^2)\frac{d}{dx}(x^2)\\ &= \frac{1}{2}\sin(x^2)(2x) = x\sin(x^2).\end{aligned}$$

Example 7.8 Find

$$\int \frac{3x}{(x^2+3)^4}\,dx.$$

Example 7.9 Find $\int \cos^2(x)\sin(x)\,dx$.

Example 7.10 Find

$$\int \frac{x^2}{(x^2+1)^2}\,dx.$$

Integration by parts

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}(uv) - \frac{\mathrm{d}u}{\mathrm{d}x}v = u\frac{\mathrm{d}v}{\mathrm{d}x}$$

(subtracting $(\mathrm{d}u/\mathrm{d}x)\ v$ from both sides)

$$\Leftrightarrow u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - \frac{\mathrm{d}u}{\mathrm{d}x}v$$

$$\Rightarrow \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \quad \text{(integrating both sides)}$$

$$\boxed{\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.}$$

Example 7.11 Find $\int x \sin x\,\mathrm{d}x$

Solution Use $u = x$; $\mathrm{d}v = \sin(x)\,\mathrm{d}x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad \text{and} \quad v = \int \sin x\,\mathrm{d}x = -\cos(x).$$

Substitute in $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$ to give

$$\begin{aligned}\int x \sin x\,\mathrm{d}x &= -x\cos(x) - \int -\cos(x)1\,\mathrm{d}x\\ &= -x\cos(x) + \sin(x) + C.\end{aligned}$$

Check:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(-x\cos(x) + \sin(x) + C) &= -\cos(x) + x\sin(x) + \cos(x)\\ &= x\sin(x).\end{aligned}$$

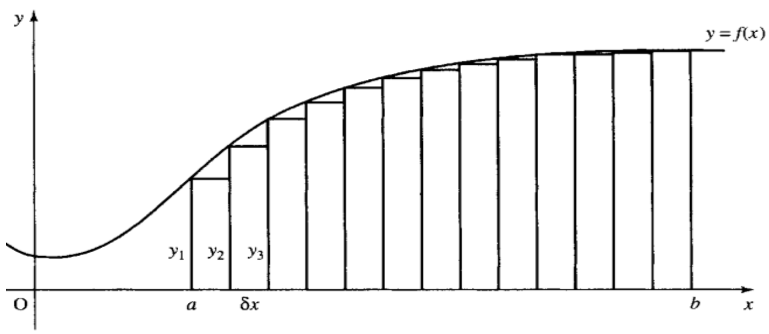
Example 7.12 Find

$$\int \frac{x^2}{(x^2+1)^2}\,\mathrm{d}x.$$

The Definite Integral

- The definite integral from (**x=a to x = b)** is defined as the area under the curve between those two points.
- In the graph in Figure below, the area under the graph has been approximated by dividing it into rectangles.
- The height of each is the value of **y** and if each rectangle is the same width then the area of the rectangle is (**yδx)**.
- If the rectangle is very thin, then **y** will not vary very much over its width and the area can logically be approximated as the sum of all of these rectangles.

$$A = y_1\delta x + y_2\delta x + y_3\delta x + y_4\delta x + \cdots = \sum_{x=a}^{x=b-\delta x} y\delta x.$$



When δx=0.1, the approximate calculation gives

1×0.1+1.1×0.1+1.2×0.1+1.3×0.1+1.4×0.1+1.5×0.1+1.6×0.1+1.7×0.1+1.8×0.1+1.9×0.1= 1.45

When δx =0.01, the calculation gives

1×0.01+1.01×0.01+1.02×0.01+ ⋯+1.98×0.01+1.99×0.01=1.495

When δx=0.001, the calculation gives

1×0.001+1.001×0.001+1.002×0.001+ ⋯+1.998×0.001+1.999×0.001=1.4995

✓ ***The area under the curve, y=f(x) between (x=a and x= b) is found as:***

$$\int_a^b y\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{x=b-\delta x} y\,\delta x$$

- ✓ The definite integral of **y** from **x = a to x = b** equals the limit as δx tends to 0 of the sum of y times δx for all x from x = a to x = b−δx.
- ✓ This is the definition of the definite integral which gives a number asits result, not a function.

Example 7.17 Find $\int_2^3 2t\,dt$.

This is the area under the graph from $t = 2$ to $t = 3$. As $\int 2t\,dt = t^2 + C$, the area up to 2 is $(2)^2 + C = 4 + C$ and the area up to 3 is $(3)^2 + C = 9 + C$. The difference in the areas is $9 + C - (4 + C) = 9 - 4 = 5$. Therefore, $\int_2^3 2t\,dt = 5$.

The working of a definite integral is usually laid out as follows

$$\int_2^3 2t\,dt = \left[t^2\right]_2^3 = (3)^2 - (2)^2 = 5.$$

Example 7.18 Find

$$\int_{-1}^1 3x^2 + 2x - 1\,dx.$$

Solution

$$\begin{aligned}\int_{-1}^1 3x^2 + 2x - 1\,dx &= \left[x^3 + x^2 - x\right]_{-1}^1 \\ &= (1^3 + 1^2 - 1) - ((-1)^3 + (-1)^2 - (-1)) \\ &= 1 - (-1 + 1 + 1) = 1 - 1 = 0.\end{aligned}$$

,

Example 7.19 Find

$$\int_0^{\pi/6} \sin(3x + 2)\,dx.$$

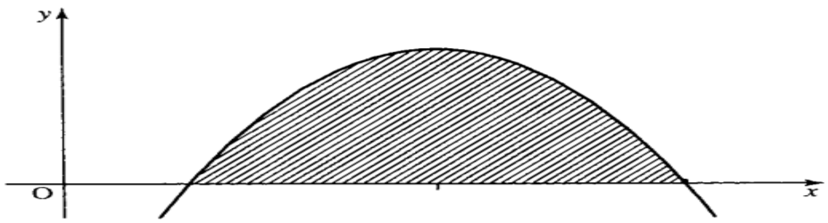
Solution

$$\begin{aligned}\int_0^{\pi/6} \sin(3x + 2)\,dx &= \left[-\frac{1}{3}\cos(3x + 2)\right]_0^{\pi/6} \\ &= \frac{1}{3}\cos\left(3\frac{\pi}{6} + 2\right) - \left(-\frac{1}{3}\cos(2)\right) \\ &= \frac{1}{3}\cos\left(\frac{\pi}{2} + 2\right) + \frac{1}{3}\cos(2) \approx 0.1644.\end{aligned}$$

Example 7.20 Find the shaded area in Figure 7.6, where $y = -x^2 + 6x - 5$.

Solution First, we find where the curve crosses the x-axis, that is, when $y = 0$

$$0 = -x^2 + 6x - 5 \quad \Leftrightarrow \quad x^2 - 6x + 5 = 0$$
$$\Leftrightarrow \quad (x-5)(x-1) = 0 \quad \Leftrightarrow \quad x = 5 \vee x = 1$$



This has given the limits of the integration. Now we integrate:

$$\begin{aligned}\int_1^5 -x^2 + 6x - 5\,\mathrm{d}x &= \left[-\frac{x^3}{3} + \frac{6x^2}{2} - 5x\right]_1^5 \\ &= -\frac{(5)^3}{3} + \frac{6(5)^2}{2} - 5(5) \\ &\quad - \left(-\frac{(1)^3}{3} + \frac{6(1)^2}{2} - 5(1)\right) \\ &= -\frac{125}{3} + 75 - 25 + \frac{1}{3} - 3 + 5 = \frac{32}{3} = 10\tfrac{2}{3}\end{aligned}$$

Therefore, the shaded area is $10\frac{2}{3}$ units^2.

TABLE 5.3 Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — Also a Definition
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any Number k
   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then
$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$
7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$
   $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)

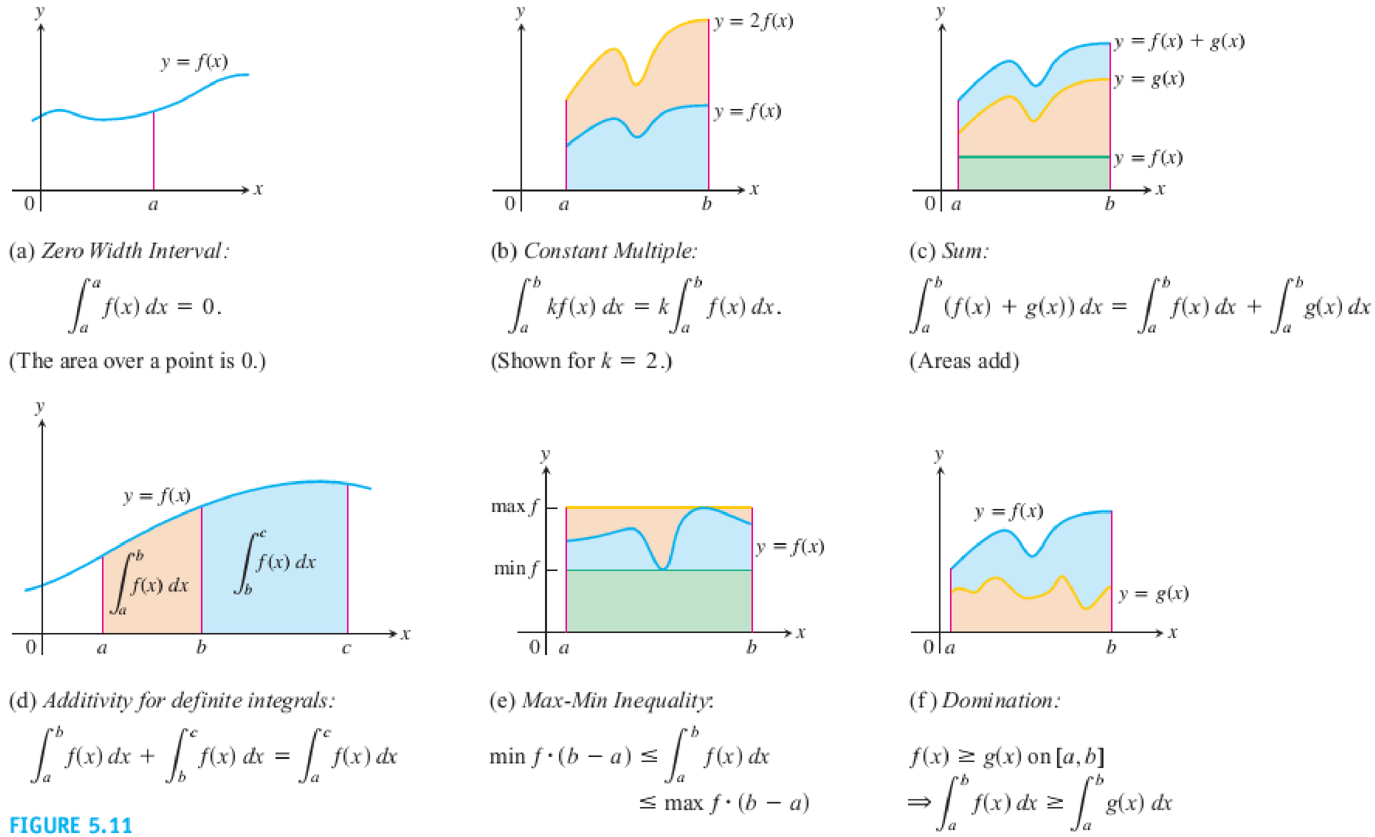
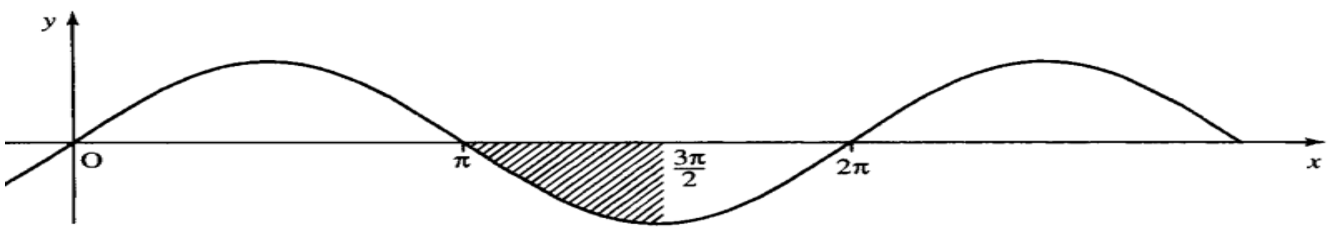


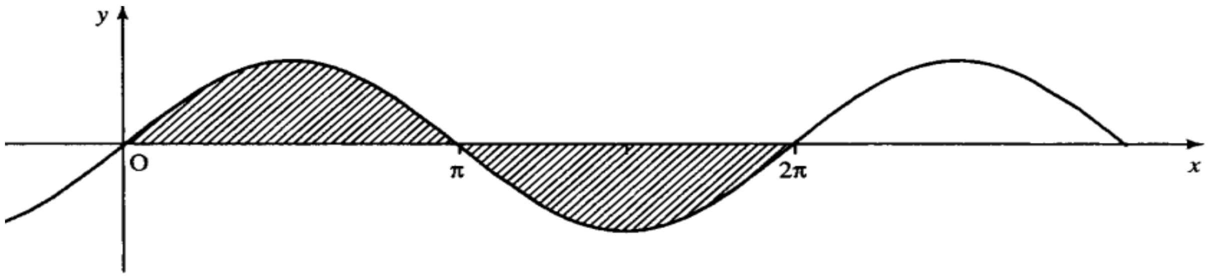
FIGURE 5.11

Example: Find the negative area. y =sin(x) from x =π to x= 3π/2 .



$$\int_{\pi}^{3\pi/2} \sin(x)\,dx = [-\cos(x)]_{\pi}^{3\pi/2} = -\cos\left(\frac{3\pi}{2}\right) + \cos(\pi) = -1$$

Example: Find the negative area. y =sin(x) from x =0 to 2π



The area under the graph $y = \sin(x)$ *from* $x = 0$ *to* 2π.

$$\int_0^{2\pi} \sin(x)\,dx = [-\cos(x)]_0^{2\pi} = -\cos(2\pi) - (-\cos(0))$$
$$= -1 - (-1) = 0$$

- To prevent cancellation of the positive and negative parts of the integration.
- we find the total shaded area in two stages.

$$\int_0^{\pi} \sin(x)\,dx = [-\cos(x)]_0^{\pi} = -\cos(\pi) - (-\cos(0)) = 2$$

and

$$\int_{\pi}^{2\pi} \sin(x)\,dx = [-\cos(x)]_{\pi}^{2\pi} = -\cos(2\pi) - (-\cos(\pi)) = -2$$

So, the total area is $2 + |-2| = 4$.

Example:

Find the area bounded by the curve $y=x^2-x$ and the x-axis and the lines $x=-1$ and $x=1$.

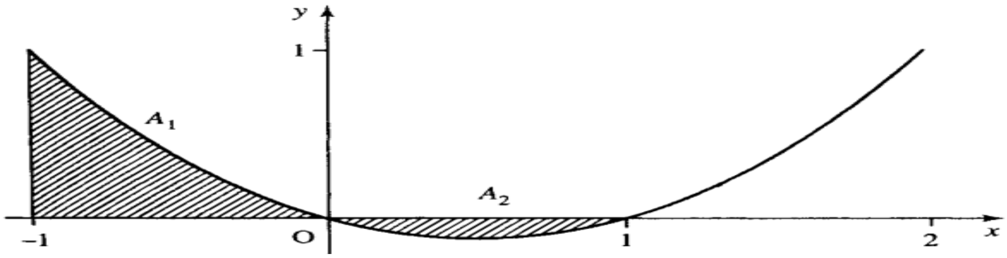
Solution First, we find if the curve crosses thex-axis.$x^2 - x = 0$
$x(x-1) = 0 \Leftrightarrow x= 0$ or $x =1$.

The sketch of the graph with the required area shaded is given in Figure below. Therefore, the area is the sum ofA1and A2.

- We find A1 by integrating from −1 to 0.

$$\int_{-1}^{0} (x^2 - x)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{0} = 0 - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right)$$
$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

$$A_1 = \frac{5}{6}.$$



Find A_2 by integrating from 0 to 1 and taking the modulus

$$\int_0^1 (x^2 - x)\,\mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

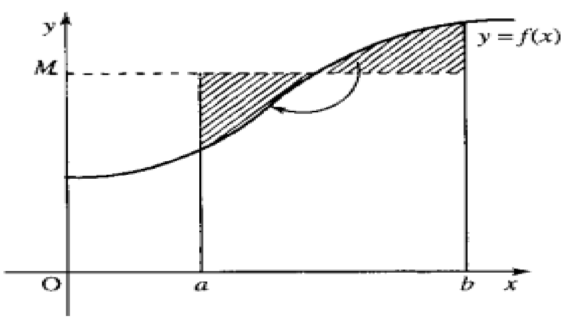
Therefore, $A_2 = \frac{1}{6}$.

Then, the total area is $A_1 + A_2 = \frac{5}{6} + \frac{1}{6} = 1$.

THE MEAN VALUE AND R.M.S. VALUE

The mean value of a function is the value it would have take if it were constant over the range but with the same area under the graph, that is, with the same integral as shown.

The formula for the mean value is



$$M = \frac{1}{b-a}\int_a^b y\,\mathrm{d}x.$$

Example 7.22 Find the mean value of $i(t) = 20 + 2\sin(\pi t)$ for $t = 0$ to 0.5.

Solution Using the formula $a = 0, b = 0.5$ gives

$$M = \frac{1}{0.5 - 0}\int_0^{0.5} 20 + 2\sin(\pi t)\,\mathrm{d}t$$

$$2\left[20t - \frac{2}{\pi}\cos(\pi t)\right]_0^{0.5} = 2(10 - 0 - \left(0 - \frac{2}{\pi}(1)\right) \approx 21.27$$

THE ROOT MEAN SQUARED (R.M.S) VALUE

The (r.m.s. value) means the square root of the mean value of the square of y. The formula for the r.m.s. value of y between x=a and x=b is.

$$\text{r.m.s.}(y) = \sqrt{\frac{1}{b-a}\int_a^b y^2\,dx}$$

- The advantage of the r.m.s. value is that as all the values for y are squared, they are positive, so the r.m.s. value will not give 0 unless we are considering the zero function.

Example 7.23 Find the r.m.s. value of $y = x^2 - 3$ between $x = 1$ and $x = 3$.

$$\begin{aligned}(\text{r.m.s.}(y))^2 &= \frac{1}{3-1}\int_1^3 (x^2-3)^2\,dx = \frac{1}{2}\int_1^3 (x^4 - 6x^2 + 9)\,dx\\ &= \frac{1}{2}\left[\frac{x^5}{5} - \frac{6x^3}{3} + 9x\right]_1^3\\ &= \frac{1}{2}\left(\left(\frac{243}{5} - 54 + 27\right) - \left(\frac{1}{5} - 2 + 9\right)\right) = 7.2\end{aligned}$$

Therefore, the r.m.s value is $\sqrt{7.2} \approx 2.683$.

EXERCISES 5.3

Using Area to Evaluate Definite Integrals

In Exercises 15–22, graph the integrands and use areas to evaluate the integrals.

15. $\int_{-2}^{4}\left(\frac{x}{2} + 3\right)dx$ 16. $\int_{1/2}^{3/2}(-2x + 4)\,dx$

17. $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$
18. $\int_{-4}^{0} \sqrt{16 - x^2}\, dx$
19. $\int_{-2}^{1} |x|\, dx$
20. $\int_{-1}^{1} (1 - |x|)\, dx$
21. $\int_{-1}^{1} (2 - |x|)\, dx$
22. $\int_{-1}^{1} (1 + \sqrt{1 - x^2})\, dx$

Use areas to evaluate the integrals in Exercises 23–26.

23. $\int_{0}^{b} \frac{x}{2}\, dx, \quad b > 0$
24. $\int_{0}^{b} 4x\, dx, \quad b > 0$
25. $\int_{a}^{b} 2s\, ds, \quad 0 < a < b$
26. $\int_{a}^{b} 3t\, dt, \quad 0 < a < b$

Evaluations

Use the results of Equations (1) and (3) to evaluate the integrals in Exercises 27–38.

27. $\int_{1}^{\sqrt{2}} x\, dx$
28. $\int_{0.5}^{2.5} x\, dx$
29. $\int_{\pi}^{2\pi} \theta\, d\theta$
30. $\int_{\sqrt{2}}^{5\sqrt{2}} r\, dr$
31. $\int_{0}^{\sqrt[3]{7}} x^2\, dx$
32. $\int_{0}^{0.3} s^2\, ds$
33. $\int_{0}^{1/2} t^2\, dt$
34. $\int_{0}^{\pi/2} \theta^2\, d\theta$
35. $\int_{a}^{2a} x\, dx$
36. $\int_{a}^{\sqrt{3}a} x\, dx$
37. $\int_{0}^{\sqrt[3]{b}} x^2\, dx$
38. $\int_{0}^{3b} x^2\, dx$

Use the rules in Table 5.3 and Equations (1)–(3) to evaluate the integrals in Exercises 39–50.

39. $\int_{3}^{1} 7\, dx$
40. $\int_{0}^{-2} \sqrt{2}\, dx$
41. $\int_{0}^{2} 5x\, dx$
42. $\int_{3}^{5} \frac{x}{8}\, dx$
43. $\int_{0}^{2} (2t - 3)\, dt$
44. $\int_{0}^{\sqrt{2}} (t - \sqrt{2})\, dt$
45. $\int_{2}^{1} \left(1 + \frac{z}{2}\right) dz$
46. $\int_{3}^{0} (2z - 3)\, dz$
47. $\int_{1}^{2} 3u^2\, du$
48. $\int_{1/2}^{1} 24u^2\, du$
49. $\int_{0}^{2} (3x^2 + x - 5)\, dx$
50. $\int_{1}^{0} (3x^2 + x - 5)\, dx$

20.10 $\int_0^{\pi/4} \cos x\, dx$. ,,,, **20.11** $\int_0^{\pi/3} \sec^2 x\, dx$. ,,,, **20.22** $\int_0^{\pi/4} \tan x \sec^2 x\, dx$.

20.24 $\int_0^{\pi/2} \sqrt{\sin x + 1} \cos x\, dx$.

,,,,

20.32 Find the average value of $f(x) = \sqrt[3]{x}$ on $[0, 1]$

20.33 Compute the average value of $f(x) = \sec^2 x$ on $[0, \pi/4]$.

20.60 The region above the x-axis and under the curve $y = \sin x$, between $x = 0$ and $x = \pi$, is divided into two parts by the line $x = c$. If the area of the left part is one-third the area of the right part, find c.

20.61 Find the value(s) of k for which $\int_0^2 x^k\, dx = \int_0^2 (2 - x)^k\, dx$.

CHAPTER THREE

HYPERBOLIC FUNCTIONS

- Another kind of functions that play important roles in applications are hyperbolic functions.
- Used in problems such as computing the tension in a cable hanged on two poles like an electric transmission line.
- The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}.

✓ Even function f satisfies f(-x) = f(x)

✓ Odd function satisfies f(-x) = -f(x)

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd part}}.$$

If we write e^x this way, we get

$$e^x = \underbrace{\frac{e^x+e^{-x}}{2}}_{\text{even part}} + \underbrace{\frac{e^x-e^{-x}}{2}}_{\text{odd part}}.$$

The even and odd parts of e^x, called the hyperbolic cosine and hyperbolic sine of x.

The **hyperbolic cosine function**, written cosh x, is defined for all real values of x by the relation

$$\boxed{\cosh x = \tfrac{1}{2}(e^x + e^{-x})}$$

Similarly the **hyperbolic sine function**, sinh x, is defined by

$$\boxed{\sinh x = \tfrac{1}{2}(e^x - e^{-x})}$$

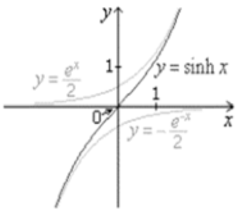


Fig. 4.1

Graph of $y = \sinh x$.

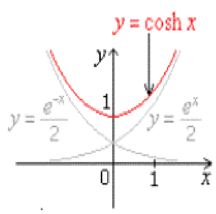


Fig. 4.2

Graph of $y = \cosh x$.

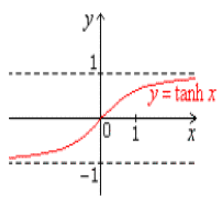


Fig. 4.3

Graph of $y = \tanh x$.

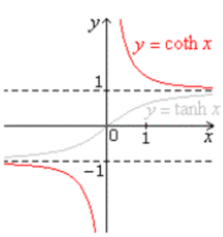


Fig. 4.4

Graph of $y = \coth x$.

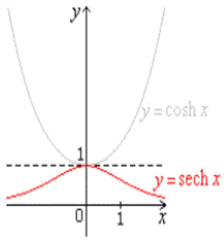


Fig. 4.5

Graph of $y = \operatorname{sech} x$.

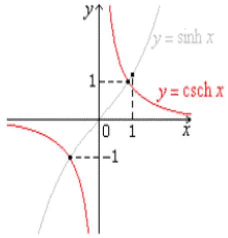


Fig. 4.6

Graph of $y = \operatorname{csch} x$.

Name	Definition	Derivative
Hyperbolic sine of x	$\sinh x = \frac{e^x - e^{-x}}{2}$	$\cosh x$
Hyperbolic cosine of x	$\cosh x = \frac{e^x + e^{-x}}{2}$	$\sinh x$
Hyperbolic tangent of x	$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	$\operatorname{sech}^2 x$
Hyperbolic cotangent of x	$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$	$-\operatorname{csch}^2 x$
Hyperbolic secant of x	$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$	$-\operatorname{sech} x \tanh x$
Hyperbolic cosecant of x	$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$	$-\operatorname{csch} x \coth x$

hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

(a) $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

(b) $\sin 3A = 3\sin A - 4\sin^3 A$

(c) $\cos^2\theta + \sin^2\theta = 1$

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \quad \text{and} \quad \operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\operatorname{sech}^2 x = 1 - \tanh^2 x.$$

1. Simplify the following expressions.

a. $\sinh \ln x$.

b. $\dfrac{\cosh \ln x + \sinh \ln x}{\cosh \ln x - \sinh \ln x}$.

Solution

a. $\sinh \ln x = \dfrac{e^{\ln x} - e^{-\ln x}}{2} = \dfrac{x - e^{\ln(1/x)}}{2} = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$

$$\boxed{\sinh(\ln x) = \tfrac{1}{2}(e^{\ln x} - e^{-\ln x}) = \tfrac{1}{2}\left(x - \tfrac{1}{x}\right)}$$

b. $\cosh \ln x = \dfrac{e^{\ln x} + e^{-\ln x}}{2} = \dfrac{1}{2}\left(x + \dfrac{1}{x}\right)$,

$$\frac{\cosh \ln x + \sinh \ln x}{\cosh \ln x - \sinh \ln x} = \frac{\frac{1}{2}\left(x+\frac{1}{x}\right) + \frac{1}{2}\left(x-\frac{1}{x}\right)}{\frac{1}{2}\left(x+\frac{1}{x}\right) - \frac{1}{2}\left(x-\frac{1}{x}\right)} = \frac{x}{\frac{2}{2x}} = x^2.$$

Ex/ Find X value ?

Suppose $\sinh x = \dfrac{3}{4}$

$\cosh^2 x = 1 + \sinh^2 x$

when $\sinh x = \dfrac{3}{4}$, $\cosh x = \dfrac{5}{4}$.

$\sinh x + \cosh x = e^x$

so $\qquad e^x = \dfrac{3}{4} + \dfrac{5}{4} = 2$

and hence $\qquad x = \ln 2$.

Alternatively, we can write $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

so $\sinh x = \frac{3}{4}$ means

$$\frac{1}{2}\left(e^x - e^{-x}\right) = \frac{3}{4}$$

$$\Rightarrow \qquad 2e^x - 3 - 2e^{-x} = 0$$

and multiplying by e^x

$$2e^{2x} - 3e^x - 2 = 0$$

$$\left(e^x - 2\right)\left(2e^x + 1\right) = 0$$

$$e^x = 2 \quad \text{or} \quad e^x = -\frac{1}{2}$$

But e^x is always positive so $e^x = 2 \quad \Rightarrow \quad x = \ln 2$.

H.W

Find the values of x for which

$$\cosh x = \frac{13}{5}$$

expressing your answers as natural logarithms.

Example

Solve the equation

$$2\cosh 2x + 10\sinh 2x = 5$$

giving your answer in terms of a natural logarithm.

Solution

$$\cosh 2x = \frac{1}{2}\left(e^{2x} + e^{-2x}\right); \quad \sinh 2x = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$$

So $$e^{2x} + e^{-2x} + 5e^{2x} - 5e^{-2x} = 5$$

$$6e^{2x} - 5 - 4e^{-2x} = 0 \qquad \boxed{*e^{\wedge}2x}$$

$$6e^{4x} - 5e^{2x} - 4 = 0$$

$$\left(3e^{2x} - 4\right)\left(2e^{2x} + 1\right) = 0$$

$$e^{2x} = \frac{4}{3} \quad \text{or} \quad e^{2x} = -\frac{1}{2}$$

The only real solution occurs when $e^{2x} > 0$

So $$2x = \ln\frac{4}{3} \quad \Rightarrow \quad x = \frac{1}{2}\ln\frac{4}{3}$$

Example

Prove that $\cosh(x-y) = \cosh x\cosh y - \sinh x\sinh y$

Solution

$$\cosh x\ \cosh y = \frac{1}{2}\left(e^{x} + e^{-x}\right) \times \frac{1}{2}\left(e^{y} + e^{-y}\right)$$

$$= \frac{1}{4}\left(e^{x+y} + e^{x-y} + e^{-(x-y)} + e^{-(x+y)}\right)$$

$$\sinh x\sinh y = \frac{1}{2}\left(e^{x} - e^{-x}\right) \times \frac{1}{2}\left(e^{y} - e^{-y}\right)$$

$$= \frac{1}{4}\left(e^{x+y} - e^{x-y} - e^{-(x-y)} + e^{-(x+y)}\right)$$

Subtracting gives

$$\cosh x\cosh y - \sinh x\sinh y = 2 \times \frac{1}{4}\left(e^{x-y} + e^{-(x-y)}\right)$$

$$= \frac{1}{2}\left(e^{x-y} + e^{-(x-y)}\right) = \cosh(x-y)$$

Example 1.1

Simplify the expression $\tanh \ln x$.

Solution

$$\tanh \ln x = \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}} = \frac{x - e^{\ln \frac{1}{x}}}{x + e^{\ln \frac{1}{x}}} = \frac{x - \frac{1}{x}}{x + \frac{1}{x}} = \frac{\frac{x^2 - 1}{x}}{\frac{x^2 + 1}{x}} = \frac{x^2 - 1}{x^2 + 1}.$$

Example 2.1

Derive from the addition identities for $\sinh(x + y)$ and $\cosh(x + y)$ the identity:

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$

Solution

$$\begin{aligned}
\tanh(x + y) &= \frac{\sinh(x + y)}{\cosh(x + y)} \\
&= \frac{\sinh x \cosh y + \sinh y \cosh x}{\cosh x \cosh y + \sinh x \sinh y} \\
&= \frac{\frac{\sinh x \cosh y + \sinh y \cosh x}{\cosh x \cosh y}}{\frac{\cosh x \cosh y + \sinh x \sinh y}{\cosh x \cosh y}} \\
&= \frac{\frac{\sinh x}{\cosh x} + \frac{\sinh y}{\cosh y}}{1 + \frac{\sinh x}{\cosh x} \cdot \frac{\sinh y}{\cosh y}} \\
&= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.
\end{aligned}$$

Exercise 2B

1. Given that $\sinh x = \frac{5}{12}$, find the values of

   (a) $\cosh x$ (b) $\tanh x$ (c) $\operatorname{sech} x$

   (d) $\coth x$ (e) $\sinh 2x$ (f) $\cosh 2x$

   Determine the value of x as a natural logarithm.

2. Given that $\cosh x = \frac{5}{4}$, determine the values of

   (a) $\sinh x$ (b) $\cosh 2x$ (c) $\sinh 2x$

   Use the formula for $\cosh(2x + x)$ to determine the value of $\cosh 3x$.

$$\begin{aligned}
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y, \\
\cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y.
\end{aligned}$$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.$$

The derivative formulas are derived from the derivative of e^u:

$$\begin{aligned}
\frac{d}{dx}(\sinh u) &= \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) && \text{Definition of } \sinh u \\
&= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2} && \text{Derivative of } e^u \\
&= \cosh u \frac{du}{dx} && \text{Definition of } \cosh u
\end{aligned}$$

This gives the first derivative formula. The calculation

$$\begin{aligned}
\frac{d}{dx}(\operatorname{csch} u) &= \frac{d}{dx}\left(\frac{1}{\sinh u}\right) && \text{Definition of } \operatorname{csch} u \\
&= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} && \text{Quotient Rule} \\
&= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} && \text{Rearrange terms.} \\
&= -\operatorname{csch} u \coth u \frac{du}{dx} && \text{Definitions of } \operatorname{csch} u \text{ and } \coth u
\end{aligned}$$

Example

Differentiate

(a) $\cosh^{-1}(2x+1)$ (b) $\sinh^{-1}\left(\frac{1}{x}\right)$ with respect to x $(x>0)$.

Solution

(a) Use the function of a function or chain rule.

$$\frac{d}{dx}[\cosh^{-1}(2x+1)] = 2.\frac{1}{\sqrt{\{(2x+1)^2-1\}}} = \frac{2}{\sqrt{(4x^2+4x)}} = \frac{1}{\sqrt{(x^2+x)}}$$

(b) $$\frac{d}{dx}\left[\sinh^{-1}\left(\frac{1}{x}\right)\right] = \frac{-1}{x^2}.\frac{1}{\sqrt{\left\{\frac{1}{x^2}+1\right\}}} = -\frac{1}{x^2}.\frac{x}{\sqrt{(1+x^2)}}$$

$$= \frac{-1}{x\sqrt{(1+x^2)}}$$

Exercise 2E

Differentiate each of the expressions in Questions 1 to 6 with respect to x.

1. $\cosh^{-1}(4+3x)$
2. $\sinh^{-1}(\sqrt{x})$
3. $\tanh^{-1}(3x+1)$
4. $x^2\sinh^{-1}(2x)$
5. $\cosh^{-1}\left(\frac{1}{x}\right)$ $(x>0)$
6. $\sinh^{-1}(\cosh 2x)$
7. Differentiate $\operatorname{sech}^{-1}x$ with respect to x, by first writing $x = \operatorname{sech} y$.
8. Find an expression for the derivative of $\operatorname{cosech}^{-1}x$ in terms of x.
9. Prove that

$$\frac{d}{dx}(\coth^{-1}x) = \frac{-1}{(x^2-1)}.$$

Example

Integrate each of the following with respect to x.

(a) $\cosh 3x$ (b) $\sinh^2 x$

(c) $x\sinh x$ (d) $e^x \cosh x$

Solution

(a) $\int \cosh 3x\,dx = \frac{1}{3}\sinh 3x + \text{constant}$

(b) $\sinh^2 x\,dx$ can be found by using $\cosh 2x = 1 + 2\sinh^2 x$ giving

$$\frac{1}{2}\int (\cosh 2x - 1)\,dx$$

$$= \frac{1}{4}\sinh 2x - \frac{1}{2}x + \text{constant}$$

Alternatively, you could change to exponentials, giving

$$\sinh^2 x = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$$

$$\int \sinh^2 x\,dx = \frac{1}{8}e^{2x} - \frac{1}{2}x - \frac{1}{8}e^{-2x} + \text{constant}$$

Can you show this answer is identical to the one found earlier?

(c) Using integration by parts,

$$\int x\sinh x\,dx = x\cosh x - \int \cosh x\,dx$$

$$= x\cosh x - \sinh x + \text{constant}$$

(d) Certainly this is found most easily by converting to exponentials, giving

$$e^x \cosh x = \frac{1}{2}e^{2x} + \frac{1}{2}$$

$$\int e^x \cosh x\,dx = \frac{1}{4}e^{2x} + \frac{1}{2}x + \text{constant}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

EXAMPLE 1 Finding Derivatives and Integrals

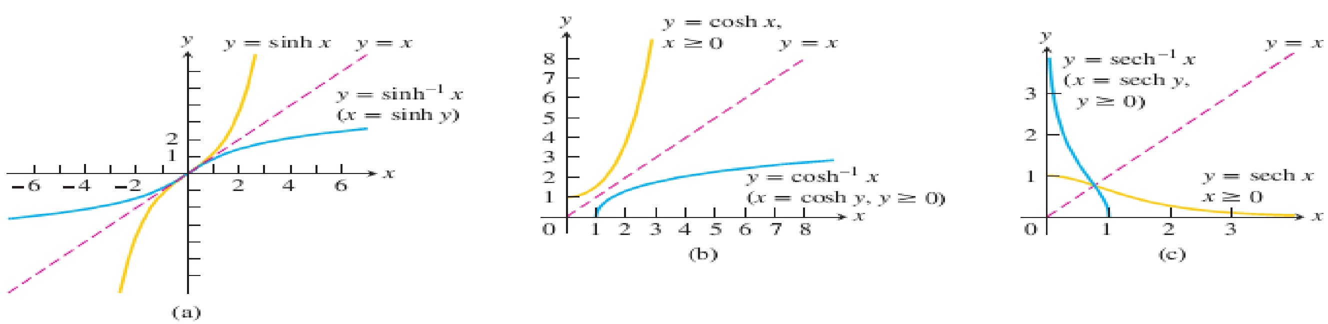
(a) $$\frac{d}{dt}\left(\tanh \sqrt{1 + t^2}\right) = \operatorname{sech}^2 \sqrt{1 = t^2} \cdot \frac{d}{dt}\left(\sqrt{1 + t^2}\right)$$
$$= \frac{t}{\sqrt{1 + t^2}} \operatorname{sech}^2 \sqrt{1 + t^2}$$

(b) $$\int \coth 5x \, dx = \int \frac{\cosh 5x}{\sinh 5x} dx = \frac{1}{5}\int \frac{du}{u}$$ $u = \sinh 5x$, $du = 5 \cosh 5x \, dx$
$$= \frac{1}{5} \ln |u| + C = \frac{1}{5} \ln |\sinh 5x| + C$$

(c) $$\int_0^1 \sinh^2 x \, dx = \int_0^1 \frac{\cosh 2x - 1}{2} dx$$ Table 7.6
$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1) \, dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$
$$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$ Evaluate with a calculator

(d) $$\int_0^{\ln 2} 4e^x \sinh x \, dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2} dx = \int_0^{\ln 2} (2e^{2x} - 2) \, dx$$
$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2 \ln 2} - 2 \ln 2) - (1 - 0)$$
$$= 4 - 2 \ln 2 - 1$$
$$\approx 1.6137$$

THE INVERSE HYPERBOLIC FUNCTIONS



$y = \sinh^{-1} x. \quad -\infty < x < \infty$

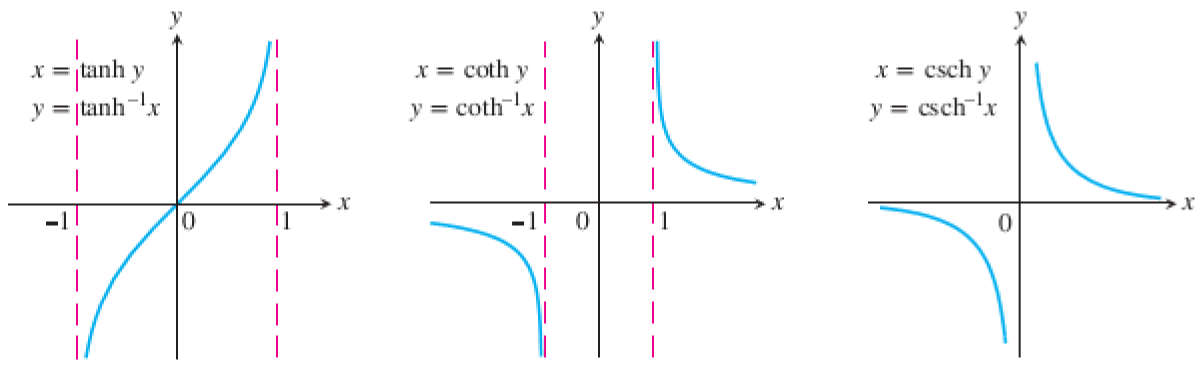
$y = \cosh x, x \geq 0, \quad y = \cosh^{-1} x. \quad 0 \leq y < \infty$

$y = \text{sech}^{-1} x.$

For every value of x in the interval $(0, 1]$, $y = \text{sech}^{-1} x$ is the nonnegative number.

❖ The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses,

$y = \tanh^{-1} x, \quad y = \coth^{-1} x, \quad y = \text{csch}^{-1} x.$



(a) (b) (c)

TABLE 7.10 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\text{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}, \qquad 0 < u < 1$$

$$\frac{d(\text{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1+u^2}}, \qquad u \neq 0$$

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

EXAMPLE 2 Derivative of the Inverse Hyperbolic Cosine

Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

Solution First we find the derivative of $y = \cosh^{-1} x$ for $x > 1$ by applying Theorem 1 with $f(x) = \cosh x$ and $f^{-1}(x) = \cosh^{-1} x$. Theorem 1 can be applied because the derivative of $\cosh x$ is positive for $0 < x$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \text{Theorem 1}$$

$$= \frac{1}{\sinh(\cosh^{-1} x)} \qquad f'(u) = \sinh u$$

$$= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} \qquad \cosh^2 u - \sinh^2 u = 1, \; \sinh u = \sqrt{\cosh^2 u - 1}$$

$$= \frac{1}{\sqrt{x^2 - 1}} \qquad \cosh(\cosh^{-1} x) = x$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}.$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}.$$

Solution The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} \qquad u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.11}$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Big]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

Hyperbolic Function Values and Identities

Each of Exercises 1–4 gives a value of sinh x or cosh x. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\frac{3}{4}$

2. $\sinh x = \frac{4}{3}$

3. $\cosh x = \frac{17}{15}, \quad x > 0$

4. $\cosh x = \frac{13}{5}, \quad x > 0$

Rewrite the expressions in Exercises 5–10 in terms of exponentials and simplify the results as much as you can.

5. $2\cosh(\ln x)$

6. $\sinh(2\ln x)$

7. $\cosh 5x + \sinh 5x$

8. $\cosh 3x - \sinh 3x$

9. $(\sinh x + \cosh x)^4$

10. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

11. Use the identities

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

to show that

a. $\sinh 2x = 2\sinh x \cosh x$

b. $\cosh 2x = \cosh^2 x + \sinh^2 x$.

12. Use the definitions of cosh x and sinh x to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

Derivatives

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

13. $y = 6\sinh\frac{x}{3}$

14. $y = \frac{1}{2}\sinh(2x + 1)$

15. $y = 2\sqrt{t}\tanh\sqrt{t}$

16. $y = t^2\tanh\frac{1}{t}$

17. $y = \ln(\sinh z)$

18. $y = \ln(\cosh z)$

19. $y = \operatorname{sech}\theta(1 - \ln\operatorname{sech}\theta)$

20. $y = \operatorname{csch}\theta(1 - \ln\operatorname{csch}\theta)$

21. $y = \ln\cosh v - \frac{1}{2}\tanh^2 v$

22. $y = \ln\sinh v - \frac{1}{2}\coth^2 v$

23. $y = (x^2 + 1)\operatorname{sech}(\ln x)$

(*Hint:* Before differentiating, express in terms of exponentials and simplify.)

24. $y = (4x^2 - 1)\operatorname{csch}(\ln 2x)$

In Exercises 25–36, find the derivative of y with respect to the appropriate variable.

25. $y = \sinh^{-1}\sqrt{x}$

26. $y = \cosh^{-1} 2\sqrt{x+1}$

27. $y = (1-\theta)\tanh^{-1}\theta$

28. $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta + 1)$

29. $y = (1-t)\coth^{-1}\sqrt{t}$

30. $y = (1-t^2)\coth^{-1} t$

31. $y = \cos^{-1} x - x\,\text{sech}^{-1} x$

32. $y = \ln x + \sqrt{1-x^2}\,\text{sech}^{-1} x$

33. $y = \text{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$

34. $y = \text{csch}^{-1} 2^{\theta}$

35. $y = \sinh^{-1}(\tan x)$

36. $y = \cosh^{-1}(\sec x), \quad 0 < x < \pi/2$

Integration Formulas

Verify the integration formulas in Exercises 37–40.

37. a. $\int \text{sech}\, x\, dx = \tan^{-1}(\sinh x) + C$

b. $\int \text{sech}\, x\, dx = \sin^{-1}(\tanh x) + C$

38. $\int x\,\text{sech}^{-1} x\, dx = \frac{x^2}{2}\text{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$

39. $\int x\coth^{-1} x\, dx = \frac{x^2-1}{2}\coth^{-1} x + \frac{x}{2} + C$

40. $\int \tanh^{-1} x\, dx = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + C$

Indefinite Integrals

Evaluate the integrals in Exercises 41–50.

41. $\int \sinh 2x\, dx$

42. $\int \sinh \frac{x}{5}\, dx$

43. $\int 6\cosh\left(\frac{x}{2} - \ln 3\right) dx$

44. $\int 4\cosh(3x - \ln 2)\, dx$

45. $\int \tanh \frac{x}{7}\, dx$

46. $\int \coth \frac{\theta}{\sqrt{3}}\, d\theta$

47. $\int \text{sech}^2\left(x - \frac{1}{2}\right) dx$

48. $\int \text{csch}^2(5 - x)\, dx$

49. $\int \frac{\text{sech}\sqrt{t}\tanh\sqrt{t}\, dt}{\sqrt{t}}$

50. $\int \frac{\text{csch}(\ln t)\coth(\ln t)\, dt}{t}$

Definite Integrals

Evaluate the integrals in Exercises 51–60.

51. $\int_{\ln 2}^{\ln 4} \coth x\, dx$

52. $\int_0^{\ln 2} \tanh 2x\, dx$

53. $\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\, d\theta$

54. $\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\, d\theta$

55. $\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\, d\theta$

56. $\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\, d\theta$

57. $\int_1^2 \frac{\cosh(\ln t)}{t}\, dt$

58. $\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\, dx$

59. $\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right) dx$

60. $\int_0^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right) dx$

SOLVED QUESTIONS CH3 // HYPERBOLIC FUNCTIONS

Hyperbolic Function Values and Identities

Each of Exercises 1–4 gives a value of $\sinh x$ or $\cosh x$. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\frac{3}{4}$ **2.** $\sinh x = \frac{4}{3}$

1. $\sinh x = -\frac{3}{4} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\left(-\frac{3}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(-\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} = -\frac{3}{5}$, $\coth x = \frac{1}{\tanh x} = -\frac{5}{3}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{4}{5}$, and $\operatorname{csch} x = \frac{1}{\sin x} = -\frac{4}{3}$

2. $\sinh x = \frac{4}{3} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{3}\right)} = \frac{4}{5}$, $\coth x = \frac{1}{\tanh x} = \frac{5}{4}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{3}{5}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{3}{4}$

12. Use the definitions of $\cosh x$ and $\sinh x$ to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

12. $\cosh^2 x - \sinh^2 x = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{1}{4}\left[(e^x+e^{-x})+(e^x-e^{-x})\right]\left[(e^x+e^{-x})-(e^x-e^{-x})\right]$
$= \frac{1}{4}(2e^x)(2e^{-x}) = \frac{1}{4}(4e^0) = \frac{1}{4}(4) = 1$

Derivatives

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

13. $y = 6\sinh\frac{x}{3}$ **14.** $y = \frac{1}{2}\sinh(2x+1)$

13. $y = 6\sinh\frac{x}{3} \Rightarrow \frac{dy}{dx} = 6\left(\cosh\frac{x}{3}\right)\left(\frac{1}{3}\right) = 2\cosh\frac{x}{3}$

14. $y = \frac{1}{2}\sinh(2x+1) \Rightarrow \frac{dy}{dx} = \frac{1}{2}[\cosh(2x+1)](2) = \cosh(2x+1)$

In Exercises 25–36, find the derivative of y with respect to the appropriate variable.

25. $y = \sinh^{-1}\sqrt{x}$ **26.** $y = \cosh^{-1} 2\sqrt{x+1}$

25. $y = \sinh^{-1}\sqrt{x} = \sinh^{-1}\left(x^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{1+(x^{1/2})^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}} = \frac{1}{2\sqrt{x(1+x)}}$

26. $y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1}\left(2(x+1)^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{(2)\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{\left[2(x+1)^{1/2}\right]^2 - 1}} = \frac{1}{\sqrt{x+1}\sqrt{4x+3}} = \frac{1}{\sqrt{4x^2+7x+3}}$

Integration Formulas

Verify the integration formulas in Exercises 37–40.

37. a. $\int \operatorname{sech} x\, dx = \tan^{-1}(\sinh x) + C$

b. $\int \operatorname{sech} x\, dx = \sin^{-1}(\tanh x) + C$

37. (a) If $y = \tan^{-1}(\sinh x) + C$, then $\frac{dy}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x$, which verifies the formula

(b) If $y = \sin^{-1}(\tanh x) + C$, then $\frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\operatorname{sech} x} = \operatorname{sech} x$, which verifies the formula

Indefinite Integrals

Evaluate the integrals in Exercises 41–50.

41. $\int \sinh 2x\, dx$ **42.** $\int \sinh \frac{x}{5}\, dx$

41. $\int \sinh 2x\, dx = \frac{1}{2}\int \sinh u\, du$, where $u = 2x$ and $du = 2\, dx$

$= \frac{\cosh u}{2} + C = \frac{\cosh 2x}{2} + C$

42. $\int \sinh \frac{x}{5}\, dx = 5\int \sinh u\, du$, where $u = \frac{x}{5}$ and $du = \frac{1}{5}\, dx$

$= 5 \cosh u + C = 5 \cosh \frac{x}{5} + C$

Definite Integrals

Evaluate the integrals in Exercises 51–60.

51. $\int_{\ln 2}^{\ln 4} \coth x\, dx$ **52.** $\int_0^{\ln 2} \tanh 2x\, dx$

51. $\int_{\ln 2}^{\ln 4} \coth x\, dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\, dx = \int_{3/4}^{15/8} \frac{1}{u}\, du = \left[\ln|u|\right]_{3/4}^{15/8} = \ln\left|\frac{15}{8}\right| - \ln\left|\frac{3}{4}\right| = \ln\left|\frac{15}{8}\cdot\frac{4}{3}\right| = \ln\frac{5}{2}$,

where $u = \sinh x$, $du = \cosh x\, dx$, the lower limit is $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2-\left(\frac{1}{2}\right)}{2} = \frac{3}{4}$ and the upper limit is $\sinh(\ln 4) = \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4-\left(\frac{1}{4}\right)}{2} = \frac{15}{8}$

52. $\int_0^{\ln 2} \tanh 2x\, dx = \int_0^{\ln 2} \frac{\sinh 2x}{\cosh 2x}\, dx = \frac{1}{2}\int_1^{17/8} \frac{1}{u}\, du = \frac{1}{2}\left[\ln|u|\right]_1^{17/8} = \frac{1}{2}\left[\ln\left(\frac{17}{8}\right) - \ln 1\right] = \frac{1}{2}\ln\frac{17}{8}$, where $u = \cosh 2x$, $du = 2\sinh(2x)\, dx$, the lower limit is $\cosh 0 = 1$ and the upper limit is $\cosh(2\ln 2) = \cosh(\ln 4)$

$= \frac{e^{\ln 4} + e^{-\ln 4}}{2} = \frac{4+\left(\frac{1}{4}\right)}{2} = \frac{17}{8}$

CHPTER FOUR

TECHNIQUES OF INTEGRATION

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \quad$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C \quad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \quad (a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \quad (u > a > 0)$

Evaluate

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx.$$

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx = \int \frac{du}{\sqrt{u}} \qquad u = x^2 - 9x + 1,\; du = (2x - 9)\,dx.$$

$$= \int u^{-1/2}\,du$$

$$= \frac{u^{(-1/2)+1}}{(-1/2) + 1} + C \qquad \text{Table 8.1 Formula 4, with } n = -1/2$$

$$= 2u^{1/2} + C$$

$$= 2\sqrt{x^2 - 9x + 1} + C$$

Evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}} \qquad a = 4,\ u = (x - 4),\ du = dx$$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 8.1, Formula 18}$$

$$= \sin^{-1}\left(\frac{x - 4}{4}\right) + C.$$

Evaluate

$$\int (\sec x + \tan x)^2\, dx.$$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2 \sec x \tan x + \tan^2 x.$$

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned} \int (\sec x + \tan x)^2\, dx &= \int (\sec^2 x + 2 \sec x \tan x + \sec^2 x - 1)\, dx \\ &= 2\int \sec^2 x\, dx + 2\int \sec x \tan x\, dx - \int 1\, dx \\ &= 2 \tan x + 2 \sec x - x + C. \end{aligned}$$

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution We use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence,

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx &= \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx \\
&= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx && \sqrt{u^2} = |u| \\
&= \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx && \text{On } [0, \pi/4], \cos 2x \geq 0, \text{ so } |\cos 2x| = \cos 2x. \\
&= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} && \text{Table 8.1, Formula 7, with } u = 2x \text{ and } du = 2\,dx \\
&= \sqrt{2}\left[\frac{1}{2} - 0\right] = \frac{\sqrt{2}}{2}.
\end{aligned}$$

Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = 3\int \frac{x\, dx}{\sqrt{1 - x^2}} + 2\int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\, dx, \qquad \text{and} \qquad x\, dx = -\frac{1}{2}du.$$

$$\begin{aligned}
3\int \frac{x\, dx}{\sqrt{1 - x^2}} &= 3\int \frac{(-1/2)\, du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\, du \\
&= -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1
\end{aligned}$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1 - x^2}} = 2\sin^{-1}x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = -3\sqrt{1 - x^2} + 2\sin^{-1}x + C.$$

Evaluate

$$\int \sec x\, dx.$$

Solution

$$\begin{aligned}
\int \sec x\, dx &= \int (\sec x)(1)\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx \\
&= \int \frac{du}{u} && u = \tan x + \sec x, \ du = (\sec^2 x + \sec x \tan x)\, dx \\
&= \ln|u| + C = \ln|\sec x + \tan x| + C.
\end{aligned}$$

INTEGRATION BY PARTS

$$\int x\,dx = \frac{1}{2}x^2 + C$$

$$\int x^2\,dx = \frac{1}{3}x^3 + C,$$

$$\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx.$$

In other words, the integral of a product is generally *not* the product of the individual-integrals:

$$\int f(x)g(x)\,dx \text{ is not equal to } \int f(x)\,dx \cdot \int g(x)\,dx.$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1 Using Integration by Parts

Find

$$\int x\cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

EXAMPLE 3 Integral of the Natural Logarithm

Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\quad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx,$ $\quad v = x.$ Simplest antiderivative

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$

EXAMPLE 4 Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

$u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Hence,

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C. \end{aligned}$$

The technique of Example 4 works for any integral $\int x^n e^x \, dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx)\right)$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

EXAMPLE 6 Finding Area

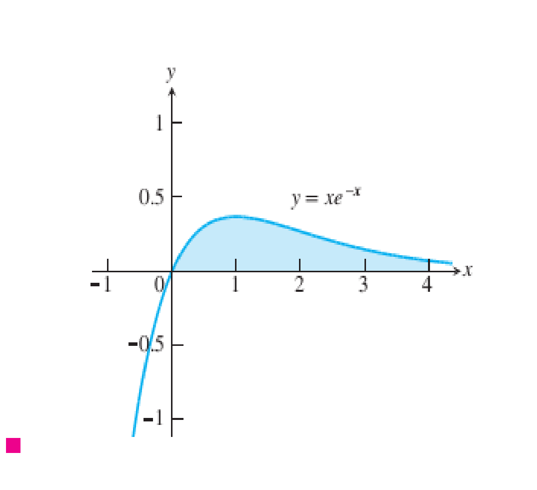
Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\,dx.$$

Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\int_0^4 xe^{-x}\,dx = -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx$$
$$= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\,dx$$
$$= -4e^{-4} - e^{-x}\Big]_0^4$$
$$= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.$$



TABULAR INTEGRATION

if we have a function in form $\int f(x)g(x)\,dx$

EXAMPLE 7 Using Tabular Integration

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

EXAMPLE 8 Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C. \qquad \blacksquare$$

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24.

1. $\int x \sin \frac{x}{2}\, dx$
2. $\int \theta \cos \pi\theta\, d\theta$
3. $\int t^2 \cos t\, dt$
4. $\int x^2 \sin x\, dx$
5. $\int_1^2 x \ln x\, dx$
6. $\int_1^e x^3 \ln x\, dx$
7. $\int \tan^{-1} y\, dy$
8. $\int \sin^{-1} y\, dy$
9. $\int x \sec^2 x\, dx$
10. $\int 4x \sec^2 2x\, dx$
11. $\int x^3 e^x\, dx$
12. $\int p^4 e^{-p}\, dp$
13. $\int (x^2 - 5x)e^x\, dx$
14. $\int (r^2 + r + 1)e^r\, dr$
15. $\int x^5 e^x\, dx$
16. $\int t^2 e^{4t}\, dt$
17. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$
18. $\int_0^{\pi/2} x^3 \cos 2x\, dx$
19. $\int_{2/\sqrt{3}}^2 t \sec^{-1} t\, dt$
20. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx$
21. $\int e^\theta \sin\theta\, d\theta$
22. $\int e^{-y} \cos y\, dy$
23. $\int e^{2x} \cos 3x\, dx$
24. $\int e^{-2x} \sin 2x\, dx$

1. $u = x,\ du = dx;\ dv = \sin\frac{x}{2}\,dx,\ v = -2\cos\frac{x}{2}$;

$$\int x \sin\frac{x}{2}\,dx = -2x\cos\frac{x}{2} - \int\left(-2\cos\frac{x}{2}\right)dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C$$

2. $u = \theta,\ du = d\theta;\ dv = \cos\pi\theta\,d\theta,\ v = \frac{1}{\pi}\sin\pi\theta$;

$$\int \theta\cos\pi\theta\,d\theta = \frac{\theta}{\pi}\sin\pi\theta - \int \frac{1}{\pi}\sin\pi\theta\,d\theta = \frac{\theta}{\pi}\sin\pi\theta + \frac{1}{\pi^2}\cos\pi\theta + C$$

Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$
26. $\int_0^1 x\sqrt{1-x}\, dx$
27. $\int_0^{\pi/3} x \tan^2 x\, dx$
28. $\int \ln(x + x^2)\, dx$
29. $\int \sin(\ln x)\, dx$
30. $\int z(\ln z)^2\, dz$

25. $\int e^{\sqrt{3s+9}}\,ds;\ \begin{bmatrix} 3s+9 = x^2 \\ ds = \frac{2}{3}x\,dx \end{bmatrix} \rightarrow \int e^x \cdot \frac{2}{3}x\,dx = \frac{2}{3}\int xe^x\,dx;\ [u = x,\ du = dx;\ dv = e^x\,dx,\ v = e^x]$;

$$\frac{2}{3}\int xe^x\,dx = \frac{2}{3}\left(xe^x - \int e^x\,dx\right) = \frac{2}{3}(xe^x - e^x) + C = \frac{2}{3}\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$$

29. $\int \sin(\ln x)\,dx;\ \begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^u\,du \end{bmatrix} \rightarrow \int(\sin u)e^u\,du.$ From Exercise 21, $\int(\sin u)e^u\,du = e^u\left(\frac{\sin u - \cos u}{2}\right) + C$

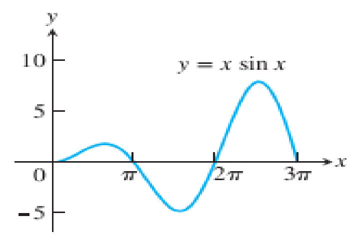
$$= \frac{1}{2}\left[-x\cos(\ln x) + x\sin(\ln x)\right] + C$$

30. $\int z(\ln z)^2\, dz;\ \begin{bmatrix} u = \ln z \\ du = \frac{1}{z}\, dz \\ dz = e^u\, du \end{bmatrix} \longrightarrow \int e^u \cdot u^2 \cdot e^u\, du = \int e^{2u} \cdot u^2\, du;$

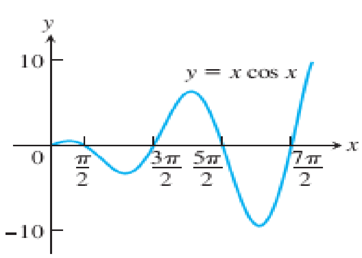
$$\begin{array}{lcl} & & e^{2u} \\ u^2 & \xrightarrow{(+)} & \frac{1}{2}e^{2u} \\ 2u & \xrightarrow{(-)} & \frac{1}{4}e^{2u} \\ 2 & \xrightarrow{(+)} & \frac{1}{8}e^{2u} \\ 0 & & \end{array}$$

$$\int u^2 e^{2u}\, du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}\left[2u^2 - 2u + 1\right] + C$$
$$= \frac{z^2}{4}\left[2(\ln z)^2 - 2\ln z + 1\right] + C$$

31. **Finding area** Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis (see the accompanying figure) for
   a. $0 \le x \le \pi$ **b.** $\pi \le x \le 2\pi$ **c.** $2\pi \le x \le 3\pi$.
   d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



32. **Finding area** Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the accompanying figure) for
   a. $\pi/2 \le x \le 3\pi/2$ **b.** $3\pi/2 \le x \le 5\pi/2$
   c. $5\pi/2 \le x \le 7\pi/2$.
   d. What pattern do you see? What is the area between the curve and the x-axis for
$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$
   n an arbitrary positive integer? Give reasons for your answer.



33. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

34. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$
   a. about the y-axis. **b.** about the line $x = 1$.

35. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about
   a. the y-axis. **b.** the line $x = \pi/2$.

36. **Finding volume** Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about
   a. the y-axis. **b.** the line $x = \pi$.
   (See Exercise 31 for a graph.)

31. (a) $u = x$, $du = dx$; $dv = \sin x\,dx$, $v = -\cos x$;

$$S_1 = \int_0^{\pi} x\sin x\,dx = [-x\cos x]_0^{\pi} + \int_0^{\pi}\cos x\,dx = \pi + [\sin x]_0^{\pi} = \pi$$

(b) $$S_2 = -\int_{\pi}^{2\pi} x\sin x\,dx = -\left[[-x\cos x]_{\pi}^{2\pi} + \int_{\pi}^{2\pi}\cos x\,dx\right] = -\left[-3\pi + [\sin x]_{\pi}^{2\pi}\right] = 3\pi$$

(c) $$S_3 = \int_{2\pi}^{3\pi} x\sin x\,dx = [-x\cos x]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi}\cos x\,dx = 5\pi + [\sin x]_{2\pi}^{3\pi} = 5\pi$$

(d) $$S_{n+1} = (-1)^{n+1}\int_{n\pi}^{(n+1)\pi} x\sin x\,dx = (-1)^{n+1}\left[[-x\cos x]_{n\pi}^{(n+1)\pi} + [\sin x]_{n\pi}^{(n+1)\pi}\right]$$
$$= (-1)^{n+1}\left[-(n+1)\pi(-1)^n + n\pi(-1)^{n+1}\right] + 0 = (2n+1)\pi$$

43. $\int \sin^{-1} x\,dx$ **44.** $\int \tan^{-1} x\,dx$

45. $\int \sec^{-1} x\,dx$ **46.** $\int \log_2 x\,dx$

43. $$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \sin y\,dy = x\sin^{-1} x + \cos y + C = x\sin^{-1} x + \cos\left(\sin^{-1} x\right) + C$$

49. $\int \sinh^{-1} x\,dx$ **50.** $\int \tanh^{-1} x\,dx$

49. (a) $$\int \sinh^{-1} x\,dx = x\sinh^{-1} x - \int \sinh y\,dy = x\sinh^{-1} x - \cosh y + C = x\sinh^{-1} x - \cosh\left(\sinh^{-1} x\right) + C;$$

check: $$d\left[x\sinh^{-1} x - \cosh\left(\sinh^{-1} x\right) + C\right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \sinh\left(\sinh^{-1} x\right)\frac{1}{\sqrt{1+x^2}}\right]dx$$
$$= \sinh^{-1} x\,dx$$

(b) $$\int \sinh^{-1} x\,dx = x\sinh^{-1} x - \int x\left(\frac{1}{\sqrt{1+x^2}}\right)dx = x\sinh^{-1} x - \frac{1}{2}\int \left(1+x^2\right)^{-1/2} 2x\,dx$$
$$= x\sinh^{-1} x - \left(1+x^2\right)^{1/2} + C$$

check: $$d\left[x\sinh^{-1} x - \left(1+x^2\right)^{1/2} + C\right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}\right]dx = \sinh^{-1} x\,dx$$

PARTIAL FRACTIONS

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called partial fractions.

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

We call

$A/(x+1)$ and $B/(x-3)$ **partial fractions**

$$5x-3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

$$A + B = 5, \qquad -3A + B = -3.$$

$$\int \frac{5x-3}{(x+1)(x-3)}dx = \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

Step 1 : Factor the denominator.

Step 2: Write fraction with one of the factors for each denominators and assign variable for each numerator.

Step 3: Multiply through out by denominator factor.

Step 4: Find the variable values.

Step 5: Write the solution as the some of two fraction.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2+px+q} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx + C_n}{(x^2+px+q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx$$

using partial fractions.

Solution The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

To find the values of the undetermined coefficients A, B, and C we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x obtaining

$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx = \int \left[\frac{3}{4}\frac{1}{x - 1} + \frac{1}{2}\frac{1}{x + 1} - \frac{1}{4}\frac{1}{x + 3}\right] dx$$

$$= \frac{3}{4}\ln|x - 1| + \frac{1}{2}\ln|x + 1| - \frac{1}{4}\ln|x + 3| + K,$$

Evaluate

$$\int \frac{6x + 7}{(x + 2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\begin{aligned} \frac{6x + 7}{(x + 2)^2} &= \frac{A}{x + 2} + \frac{B}{(x + 2)^2} \\ 6x + 7 &= A(x + 2) + B && \text{Multiply both sides by } (x + 2)^2. \\ &= Ax + (2A + B) \end{aligned}$$

Equating coefficients of corresponding powers of x gives

$A = 6$ and $2A + B = 12 + B = 7,$ or $A = 6$ and $B = -5.$

Therefore,

$$\begin{aligned} \int \frac{6x + 7}{(x + 2)^2}\,dx &= \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2}\right) dx \\ &= 6\int \frac{dx}{x + 2} - 5\int (x + 2)^{-2}\,dx \\ &= 6\ln|x + 2| + 5(x + 2)^{-1} + C \end{aligned}$$

Evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r|l} & 2x \\ \hline x^2 - 2x - 3 & 2x^3 - 4x^2 - x - 3 \\ & \underline{2x^3 - 4x^2 - 6x} \\ & \qquad\qquad 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx &= \int 2x\,dx + \int \frac{5x - 3}{x^2 - 2x - 3}\,dx \\ &= \int 2x\,dx + \int \frac{2}{x + 1}\,dx + \int \frac{3}{x - 3}\,dx \\ &= x^2 + 2\ln|x + 1| + 3\ln|x - 3| + C. \end{aligned}$$

Evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx$$

using partial fractions.

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \qquad (2)$$

$$\begin{aligned} -2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D). \end{aligned}$$

Coefficients of x^3: $\quad 0 = A + C$

Coefficients of x^2: $\quad 0 = -2A + B - C + D$

Coefficients of x^1: $\quad -2 = A - 2B + C$

Coefficients of x^0: $\quad 4 = B - C + D$

We substitute these values into Equation (2), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C. \quad ■\end{aligned}$$

Evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned}1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A\end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A=1$, $B=-1$, $C=0$, $D=-1$, and $E=0$. Thus,

$$\begin{aligned}\int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\ &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\ &= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad \begin{aligned}u &= x^2+1,\\ du &= 2x\,dx\end{aligned} \\ &= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\ &= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K. \quad ■\end{aligned}$$

EXERCISES 8.3

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x-13}{(x-3)(x-2)}$
2. $\dfrac{5x-7}{x^2-3x+2}$
3. $\dfrac{x+4}{(x+1)^2}$
4. $\dfrac{2x+2}{x^2-2x+1}$
5. $\dfrac{z+1}{z^2(z-1)}$
6. $\dfrac{z}{z^3-z^2-6z}$
7. $\dfrac{t^2+8}{t^2-5t+6}$
8. $\dfrac{t^4+9}{t^4+9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrands as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1-x^2}$
10. $\displaystyle\int \frac{dx}{x^2+2x}$
11. $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx$
12. $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$
13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$
14. $\displaystyle\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy$
15. $\displaystyle\int \frac{dt}{t^3+t^2-2t}$
16. $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

2. $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x-7 = A(x-1)+B(x-2) = (A+B)x-(A+2B)$

$$\Rightarrow \left.\begin{array}{l} A+B=5 \\ A+2B=7 \end{array}\right\} \Rightarrow B=2 \Rightarrow A=3; \text{ thus, } \frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$$

5. $\frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1)+B(z-1)+Cz^2 \Rightarrow z+1 = (A+C)z^2+(-A+B)z-B$

$$\Rightarrow \left.\begin{array}{l} A+C=0 \\ -A+B=1 \\ -B=1 \end{array}\right\} \Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2; \text{ thus, } \frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$$

9. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x)+B(1-x);\ x=1 \Rightarrow A=\frac{1}{2};\ x=-1 \Rightarrow B=\frac{1}{2};$

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}\left[\ln|1+x| - \ln|1-x|\right] + C$$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2)+Bx;\ x=0 \Rightarrow A=\frac{1}{2};\ x=-2 \Rightarrow B=-\frac{1}{2};$

$$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}\left[\ln|x| - \ln|x+2|\right] + C$$

In Exercises 17–20, express the integrands as a sum of partial fractions and evaluate the integrals.

17. $\int_0^1 \frac{x^3\,dx}{x^2+2x+1}$ **18.** $\int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$

19. $\int \frac{dx}{(x^2-1)^2}$ **20.** $\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1)+B$
$= Ax + (A+B) \Rightarrow A = 3, A+B = 2 \Rightarrow A = 3, B = -1; \int_0^1 \frac{x^3\,dx}{x^2+2x+1}$
$= \int_0^1 (x-2)\,dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$
$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1)+B$
$= Ax + (-A+B) \Rightarrow A = 3, -A+B = -2 \Rightarrow A = 3, B = 1; \int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$
$= \int_{-1}^0 (x+2)\,dx + 3\int_{-1}^0 \frac{dx}{x-1} + \int_{-1}^0 \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^0$
$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$

In Exercises 21–28, express the integrands as a sum of partial fractions and evaluate the integrals.

21. $\int_0^1 \frac{dx}{(x+1)(x^2+1)}$ **22.** $\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$

23. $\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$ **24.** $\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx$

25. $\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds$ **26.** $\int \frac{s^4+81}{s(s^2+9)^2}\,ds$

27. $\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta$

28. $\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta$

27. $\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3+5\theta^2+8\theta+4 = (A\theta+B)(\theta^2+2\theta+2)+C\theta+D$
$= A\theta^3+(2A+B)\theta^2+(2A+2B+C)\theta+(2B+D) \Rightarrow A=2; 2A+B=5 \Rightarrow B=1; 2A+2B+C=8 \Rightarrow C=2;$
$2B+D=4 \Rightarrow D=2; \int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)}\,d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2}\,d\theta$
$= \int \frac{2\theta+2}{\theta^2+2\theta+2}\,d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$

$= \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) + C$

28. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3} \Rightarrow \theta^4-4\theta^3+2\theta^2-3\theta+1$
$= (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta+F = (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3+D\theta^2+C\theta+D) + E\theta+F$
$= (A\theta^5+B\theta^4+2A\theta^3+2B\theta^2+A\theta+B) + (C\theta^3+D\theta^2+C\theta+D) + E\theta+F$
$= A\theta^5+B\theta^4+(2A+C)\theta^3+(2B+D)\theta^2+(A+C+E)\theta+(B+D+F) \Rightarrow A=0; B=1; 2A+C=-4$
$\Rightarrow C=-4; 2B+D=2 \Rightarrow D=0; A+C+E=-3 \Rightarrow E=1; B+D+F=1 \Rightarrow F=0;$
$\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta = \int \frac{d\theta}{\theta^2+1} - 4\int \frac{\theta\,d\theta}{(\theta^2+1)^2} + \int \frac{\theta\,d\theta}{(\theta^2+1)^3} = \tan^{-1}\theta + 2(\theta^2+1)^{-1} - \frac{1}{4}(\theta^2+1)^{-2} + C$

In Exercises 29–34, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

29. $\int \frac{2x^3-2x^2+1}{x^2-x}\,dx$ **30.** $\int \frac{x^4}{x^2-1}\,dx$

31. $\int \frac{9x^3-3x+1}{x^3-x^2}\,dx$ **32.** $\int \frac{16x^3}{4x^2-4x+1}\,dx$

33. $\int \frac{y^4+y^2-1}{y^3+y}\,dy$ **34.** $\int \frac{2y^4}{y^3-y^2+y-1}\,dy$

29. $\frac{2x^3-2x^2+1}{x^2-x} = 2x + \frac{1}{x^2-x} = 2x + \frac{1}{x(x-1)}; \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1)+Bx; x=0 \Rightarrow A=-1;$
$x=1 \Rightarrow B=1; \int \frac{2x^3-2x^2+1}{x^2-x} = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C$

30. $\frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = (x^2+1) + \frac{1}{(x+1)(x-1)}; \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1)+B(x+1);$
$x=-1 \Rightarrow A=-\frac{1}{2}; x=1 \Rightarrow B=\frac{1}{2}; \int \frac{x^4}{x^2-1}\,dx = \int (x^2+1)\,dx - \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$
$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

Evaluate the integrals in Exercises 35–40.

35. $\displaystyle\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$ **36.** $\displaystyle\int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt$

37. $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y+\sin y-6}$ **38.** $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta+\cos\theta-2}$

39. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{(4x^2+1)(x-2)^2}\,dx$

40. $\displaystyle\int \frac{(x+1)^2\tan^{-1}(3x)+9x^3+x}{(9x^2+1)(x+1)^2}\,dx$

35. $$\int \frac{e^t\,dt}{e^{2t}+3e^t+2} = [e^t = y]\int \frac{dy}{y^2+3y+2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln\left|\frac{y+1}{y+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$$

40. $$\int \frac{(x+1)^2\tan^{-1}(3x)+9x^3+x}{(9x^2+1)(x+1)^2}\,dx = \int \frac{\tan^{-1}(3x)}{9x^2+1}\,dx + \int \frac{x}{(x+1)^2}\,dx$$
$$= \frac{1}{3}\int \tan^{-1}(3x)\,d\left(\tan^{-1}(3x)\right) + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{\left(\tan^{-1}3x\right)^2}{6} + \ln|x+1| + \frac{1}{x+1} + C$$

TRIGONOMETRIC INTEGRALS

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \quad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \quad (2)$$

EXAMPLE 1 m is Odd

Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cos^2 x (-d(\cos x))$$

$$= \int (1 - u^2)(u^2)(-du) \qquad u = \cos x$$

$$= \int (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$$

EXAMPLE 2 *m* is Even and *n* is Odd

Evaluate

$$\int \cos^5 x\,dx.$$

Solution

$$\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx = \int (1 - \sin^2 x)^2\,d(\sin x) \qquad m = 0$$

$$= \int (1 - u^2)^2\,du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\,du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C. \qquad \blacksquare$$

EXAMPLE 3 *m* and *n* are Both Even

Evaluate

$$\int \sin^2 x \cos^4 x\,dx.$$

$$\int \sin^2 x \cos^4 x\,dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\,dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\,dx\right].$$

For the term involving $\cos^2 2x$ we use

$$\int \cos^2 2x\,dx = \frac{1}{2}\int (1 + \cos 4x)\,dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right). \qquad \text{Omitting the constant of integration until the final result}$$

For the $\cos^3 2x$ term we have

$$\int \cos^3 2x\,dx = \int (1 - \sin^2 2x)\cos 2x\,dx \qquad u = \sin 2x,\ du = 2\cos 2x\,dx$$

$$= \frac{1}{2}\int (1 - u^2)\,du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right). \qquad \text{Again omitting } C$$

Combining everything and simplifying we get

$$\int \sin^2 x \cos^4 x\,dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C. \qquad \blacksquare$$

EXAMPLE 5 Evaluate

$$\int \tan^4 x\, dx.$$

Solution

$$\begin{aligned}\int \tan^4 x\, dx &= \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx.\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x\, dx.$$

Solution We integrate by parts, using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\, dx) \\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\, dx \qquad \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \quad (3)$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x], \quad (4)$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \quad (5)$$

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$ we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

EXERCISES 8.4

Products of Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–14.

1. $\int_0^{\pi/2} \sin^5 x \, dx$
2. $\int_0^{\pi} \sin^5 \frac{x}{2} \, dx$
3. $\int_{-\pi/2}^{\pi/2} \cos^3 x \, dx$
4. $\int_0^{\pi/6} 3\cos^5 3x \, dx$
5. $\int_0^{\pi/2} \sin^7 y \, dy$
6. $\int_0^{\pi/2} 7\cos^7 t \, dt$
7. $\int_0^{\pi} 8\sin^4 x \, dx$
8. $\int_0^{1} 8\cos^4 2\pi x \, dx$
9. $\int_{-\pi/4}^{\pi/4} 16\sin^2 x \cos^2 x \, dx$
10. $\int_0^{\pi} 8\sin^4 y \cos^2 y \, dy$
11. $\int_0^{\pi/2} 35\sin^4 x \cos^3 x \, dx$
12. $\int_0^{\pi} \sin 2x \cos^2 2x \, dx$
13. $\int_0^{\pi/4} 8\cos^3 2\theta \sin 2\theta \, d\theta$
14. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$

1. $\int_0^{\pi/2} \sin^5 x \, dx = \int_0^{\pi/2} (\sin^2 x)^2 \sin x \, dx = \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x \, dx = \int_0^{\pi/2} (1 - 2\cos^2 x + \cos^4 x)\sin x \, dx$

$= \int_0^{\pi/2} \sin x \, dx - \int_0^{\pi/2} 2\cos^2 x \sin x \, dx + \int_0^{\pi/2} \cos^4 x \sin x \, dx = \left[-\cos x + 2\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right]_0^{\pi/2}$

$= (0) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right) = \frac{8}{15}$

13. $\int_0^{\pi/4} 8\cos^3 2\theta \sin 2\theta \, d\theta = \left[8\left(-\frac{1}{2}\right)\frac{\cos^4 2\theta}{4}\right]_0^{\pi/4} = [-\cos^4 2\theta]_0^{\pi/4} = (0) - (-1) = 1$

14. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta = \int_0^{\pi/2} \sin^2 2\theta (1 - \sin^2 2\theta) \cos 2\theta \, d\theta = \int_0^{\pi/2} \sin^2 2\theta \cos 2\theta \, d\theta - \int_0^{\pi/2} \sin^4 2\theta \cos 2\theta \, d\theta$
$= \left[\frac{1}{2} \cdot \frac{\sin^3 2\theta}{3} - \frac{1}{2} \cdot \frac{\sin^5 2\theta}{5}\right]_0^{\pi/2} = 0$

Evaluate the integrals in Exercises 23–32.

23. $\int_{-\pi/3}^{0} 2 \sec^3 x \, dx$ **24.** $\int e^x \sec^3 e^x \, dx$

25. $\int_0^{\pi/4} \sec^4 \theta \, d\theta$ **26.** $\int_0^{\pi/12} 3 \sec^4 3x \, dx$

27. $\int_{\pi/4}^{\pi/2} \csc^4 \theta \, d\theta$ **28.** $\int_{\pi/2}^{\pi} 3 \csc^4 \frac{\theta}{2} \, d\theta$

29. $\int_0^{\pi/4} 4 \tan^3 x \, dx$ **30.** $\int_{-\pi/4}^{\pi/4} 6 \tan^4 x \, dx$

31. $\int_{\pi/6}^{\pi/3} \cot^3 x \, dx$ **32.** $\int_{\pi/4}^{\pi/2} 8 \cot^4 t \, dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 33–38.

33. $\int_{-\pi}^{0} \sin 3x \cos 2x \, dx$ **34.** $\int_0^{\pi/2} \sin 2x \cos 3x \, dx$

35. $\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$ **36.** $\int_0^{\pi/2} \sin x \cos x \, dx$

37. $\int_0^{\pi} \cos 3x \cos 4x \, dx$ **38.** $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$

23. $\int_{-\pi/3}^{0} 2\sec^3 x \, dx$; $u = \sec x$, $du = \sec x \tan x \, dx$, $dv = \sec^2 x \, dx$, $v = \tan x$;
$\int_{-\pi/3}^{0} 2\sec^3 x \, dx = [2 \sec x \tan x]_{-\pi/3}^{0} - 2\int_{-\pi/3}^{0} \sec x \tan^2 x \, dx = 2 \cdot 1 \cdot 0 - 2 \cdot 2 \cdot \sqrt{3} - 2\int_{-\pi/3}^{0} \sec x \, (\sec^2 x - 1) dx$
$= 4\sqrt{3} - 2\int_{-\pi/3}^{0} \sec^3 x \, dx + 2\int_{-\pi/3}^{0} \sec x \, dx$; $2\int_{-\pi/3}^{0} 2\sec^3 x \, dx = 4\sqrt{3} + [2\ln|\sec x + \tan x|]_{-\pi/3}^{0}$
$2\int_{-\pi/3}^{0} 2\sec^3 x \, dx = 4\sqrt{3} + 2\ln|1 + 0| - 2\ln|2 - \sqrt{3}| = 4\sqrt{3} - 2\ln\left(2 - \sqrt{3}\right)$
$\int_{-\pi/3}^{0} 2\sec^3 x \, dx = 2\sqrt{3} - \ln\left(2 - \sqrt{3}\right)$

32. $\int_{\pi/4}^{\pi/2} 8\cot^4 t\,dt = 8\int_{\pi/4}^{\pi/2}(\csc^2 t - 1)\cot^2 t\,dt = 8\int_{\pi/4}^{\pi/2}\csc^2 t\cot^2 t\,dt - 8\int_{\pi/4}^{\pi/2}\cot^2 t\,dt$

$= -8\left[-\frac{\cot^3 t}{3}\right]_{\pi/4}^{\pi/2} - 8\int_{\pi/4}^{\pi/2}(\csc^2 t - 1)\,dt = -\frac{8}{3}(0-1) + [8\cot t]_{\pi/4}^{\pi/2} + [8t]_{\pi/4}^{\pi/2} = \frac{8}{3} + 8(0-1) + 4\pi - 2\pi = 2\pi - \frac{16}{3}$

33. $\int_{-\pi}^{0}\sin 3x\cos 2x\,dx = \frac{1}{2}\int_{-\pi}^{0}(\sin x + \sin 5x)\,dx = \frac{1}{2}\left[-\cos x - \frac{1}{5}\cos 5x\right]_{-\pi}^{0} = \frac{1}{2}\left(-1 - \frac{1}{5} - 1 - \frac{1}{5}\right) = -\frac{6}{5}$

TRIGONOMETRIC SUBSTITUTIONS

Trigonometric substitutions can be very useful in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

Consider the following right triangles

With $x = a\tan\theta$,

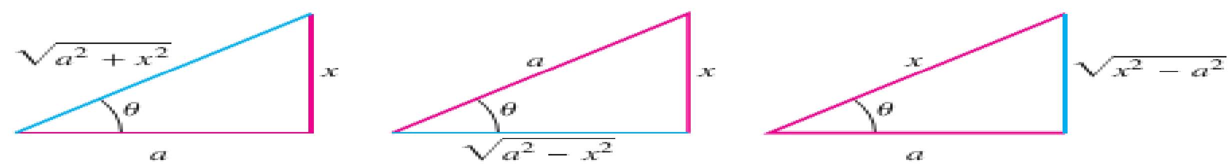
$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

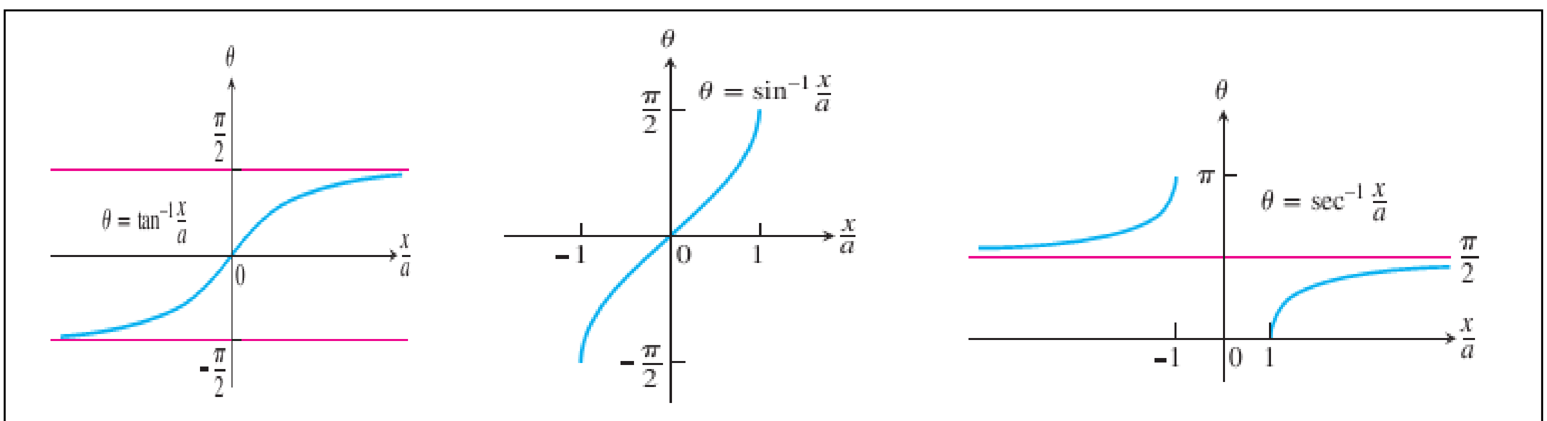
With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$



$x = a\tan\theta$ $\quad$ $x = a\sin\theta$ $\quad$ $x = a\sec\theta$

$\sqrt{a^2 + x^2} = a|\sec\theta|$ $\quad$ $\sqrt{a^2 - x^2} = a|\cos\theta|$ $\quad$ $\sqrt{x^2 - a^2} = a|\tan\theta|$



$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\ \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

EXAMPLE 1 Using the Substitution $x = a\tan\theta$

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$

Then

$$\int \frac{dx}{\sqrt{4 + x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \sqrt{\sec^2\theta} = |\sec\theta|$$

$$= \int \sec\theta\, d\theta \qquad \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4 + x^2}}{2} + \frac{x}{2}\right| + C \qquad \text{From Fig. 8.4}$$

$$= \ln|\sqrt{4 + x^2} + x| + C'. \qquad \text{Taking } C' = C - \ln 2$$

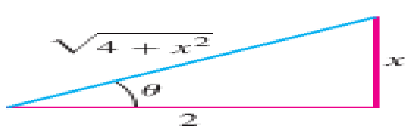


FIGURE 8.4 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4 + x^2}}{2}.$$

EXAMPLE 2 Using the Substitution $x = a\sin\theta$

Evaluate

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$

Then

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta\cdot 3\cos\theta\,d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\,d\theta \qquad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C \qquad \text{Fig. 8.5}$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

■

EXAMPLE 3 Using the Substitution $x = a\sec\theta$

Evaluate

$$\int \frac{dx}{\sqrt{25x^2-4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2-\frac{4}{25}\right)}$$
$$= 5\sqrt{x^2-\left(\frac{2}{5}\right)^2}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$

$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta. \qquad \begin{array}{l}\tan\theta > 0 \text{ for}\\ 0 < \theta < \pi/2\end{array}$$

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\,d\theta}{5\cdot(2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\,d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C. \qquad \text{Fig. 8.6}$$

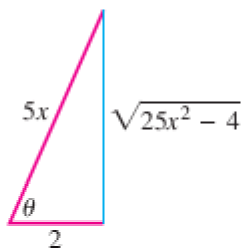


FIGURE 8.6 If $x = (2/5)\sec\theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and

EXAMPLE 5 Finding the Area of an Ellipse

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (Figure 8.9). Solving the equation of the ellipse for $y \geq 0$,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2},$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \leq x \leq a$$

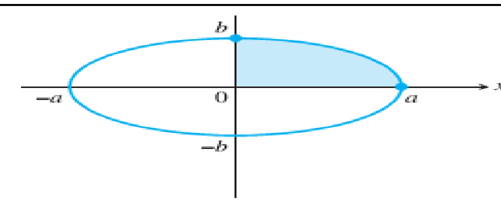


FIGURE 8.9 The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in

The area of the ellipse is

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta \qquad \begin{aligned}&x = a\sin\theta,\ dx = a\cos\theta\,d\theta,\\ &\theta = 0 \text{ when } x = 0;\\ &\theta = \pi/2 \text{ when } x = a\end{aligned}$$

$$= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\,d\theta$$

$$= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 2ab\left[\frac{\pi}{2} + 0 - 0\right] = \pi ab.$$

If we get that the area of a circle with radius **r is πr^2**

EXERCISES 8.5

Evaluate the integrals in Exercises 1–28.

1. $\int \frac{dy}{\sqrt{9 + y^2}}$
2. $\int \frac{3\,dy}{\sqrt{1 + 9y^2}}$
3. $\int_{-2}^{2} \frac{dx}{4 + x^2}$
4. $\int_0^2 \frac{dx}{8 + 2x^2}$
5. $\int_0^{3/2} \frac{dx}{\sqrt{9 - x^2}}$
6. $\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1 - 4x^2}}$
7. $\int \sqrt{25 - t^2}\,dt$
8. $\int \sqrt{1 - 9t^2}\,dt$
9. $\int \frac{dx}{\sqrt{4x^2 - 49}}, \quad x > \frac{7}{2}$
10. $\int \frac{5\,dx}{\sqrt{25x^2 - 9}}, \quad x > \frac{3}{5}$
11. $\int \frac{\sqrt{y^2 - 49}}{y}\,dy, \quad y > 7$
12. $\int \frac{\sqrt{y^2 - 25}}{y^3}\,dy, \quad y > 5$
13. $\int \frac{dx}{x^2\sqrt{x^2 - 1}}, \quad x > 1$
14. $\int \frac{2\,dx}{x^3\sqrt{x^2 - 1}}, \quad x > 1$
15. $\int \frac{x^3\,dx}{\sqrt{x^2 + 4}}$
16. $\int \frac{dx}{x^2\sqrt{x^2 + 1}}$
17. $\int \frac{8\,dw}{w^2\sqrt{4 - w^2}}$
18. $\int \frac{\sqrt{9 - w^2}}{w^2}\,dw$
19. $\int_0^{\sqrt{3}/2} \frac{4x^2\,dx}{(1 - x^2)^{3/2}}$
20. $\int_0^1 \frac{dx}{(4 - x^2)^{3/2}}$
21. $\int \frac{dx}{(x^2 - 1)^{3/2}}, \quad x > 1$
22. $\int \frac{x^2\,dx}{(x^2 - 1)^{5/2}}, \quad x > 1$

23. $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$ **24.** $\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

25. $\int \frac{8\,dx}{(4x^2+1)^2}$ **26.** $\int \frac{6\,dt}{(9t^2+1)^2}$

27. $\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$ **28.** $\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 29–36, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

29. $\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$ **30.** $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

31. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$ **32.** $\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

33. $\int \frac{dx}{x\sqrt{x^2-1}}$ **34.** $\int \frac{dx}{1+x^2}$

35. $\int \frac{x\,dx}{\sqrt{x^2-1}}$ **36.** $\int \frac{dx}{\sqrt{1-x^2}}$

1. $y = 3\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dy = \frac{3\,d\theta}{\cos^2\theta},\ 9+y^2 = 9(1+\tan^2\theta) = \frac{9}{\cos^2\theta} \Rightarrow \frac{1}{\sqrt{9+y^2}} = \frac{|\cos\theta|}{3} = \frac{\cos\theta}{3}$
(because $\cos\theta > 0$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$);
$\int \frac{dy}{\sqrt{9+y^2}} = 3\int \frac{\cos\theta\,d\theta}{3\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \ln|\sec\theta+\tan\theta| + C' = \ln\left|\frac{\sqrt{9+y^2}}{3} + \frac{y}{3}\right| + C' = \ln\left|\sqrt{9+y^2}+y\right| + C$

9. $x = \frac{7}{2}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{7}{2}\sec\theta\tan\theta\,d\theta,\ \sqrt{4x^2-49} = \sqrt{49\sec^2\theta - 49} = 7\tan\theta;$
$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{\left(\frac{7}{2}\sec\theta\tan\theta\right)d\theta}{7\tan\theta} = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C = \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C$

16. $x = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \sec^2\theta\,d\theta,\ \sqrt{x^2+1} = \sec\theta;$
$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$

26. $t = \frac{1}{3}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\sec^2\theta\,d\theta,\ 9t^2+1 = \sec^2\theta;$
$\int \frac{6\,dt}{(9t^2+1)^2} = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right)d\theta}{\sec^4\theta} = 2\int \cos^2\theta\,d\theta = \theta + \sin\theta\cos\theta + C = \tan^{-1}3t + \frac{3t}{(9t^2+1)} + C$

31. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$; $\left[u = 2\sqrt{t},\ du = \frac{1}{\sqrt{t}}\,dt\right] \rightarrow \int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2}$; $u = \tan\theta,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{4},\ du = \sec^2\theta\,d\theta,\ 1+u^2 = \sec^2\theta$;

$$\int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2} = \int_{\pi/6}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sec^2\theta} = [2\theta]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6}$$

36. $x = \sin\theta,\ dx = \cos\theta\,d\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$;

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} = \theta + C = \sin^{-1}x + C$$

IMPROPER INTEGRALS

- Definite integrals require firstly that the domain of integration [a, b] to be finite and
- Secondly that the range of the integrand to be finite over the domain.
- For example, the integral of ($\ln x/x^2$) over $[1,\infty)$, or the integral of ($1/\sqrt{x}$) over (0; 1]. Those integrals are said to be improper.

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

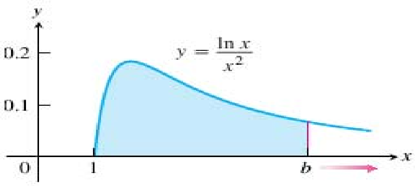


FIGURE 8.19 The area under this curve is an improper integral (Example 1).

EXAMPLE 1 Evaluating an Improper Integral on $[1, \infty)$

Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve (Figure 8.19). The area from 1 to b is

$$\begin{aligned}\int_1^b \frac{\ln x}{x^2}\,dx &= \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx \\ &= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b \\ &= -\frac{\ln b}{b} - \frac{1}{b} + 1.\end{aligned}$$

Integration by parts with $u = \ln x,\ dv = dx/x^2,\ du = dx/x,\ v = -1/x$.

The limit of the area as $b \to \infty$ is

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx \\ &= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] \\ &= -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1 \\ &= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1.\end{aligned}$$

EXAMPLE 2 Evaluating an Integral on $(-\infty, \infty)$

Evaluate

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}.$$

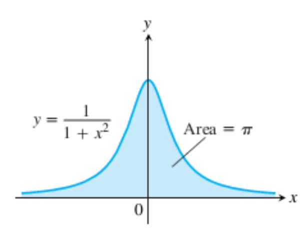
Solution According to the definition (Part 3), we can write

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{\infty}\frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}\int_{-\infty}^{0}\frac{dx}{1+x^2} &= \lim_{a\to-\infty}\int_a^0\frac{dx}{1+x^2} \\ &= \lim_{a\to-\infty}\tan^{-1}x\Big]_a^0 \\ &= \lim_{a\to-\infty}(\tan^{-1}0 - \tan^{-1}a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty}\frac{dx}{1+x^2} &= \lim_{b\to\infty}\int_0^b\frac{dx}{1+x^2} \\ &= \lim_{b\to\infty}\tan^{-1}x\Big]_0^b \\ &= \lim_{b\to\infty}(\tan^{-1}b - \tan^{-1}0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$



Thus,

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

EXERCISES 8.8

Evaluate the integrals in Exercises 1–34 without using tables.

1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$

2. $\displaystyle\int_1^\infty \frac{dx}{x^{1.001}}$

3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

4. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

5. $\displaystyle\int_{-1}^1 \frac{dx}{x^{2/3}}$

6. $\displaystyle\int_{-8}^1 \frac{dx}{x^{1/3}}$

7. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

8. $\displaystyle\int_0^1 \frac{dr}{r^{0.999}}$

9. $\displaystyle\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\displaystyle\int_{-\infty}^2 \frac{2\,dx}{x^2+4}$

11. $\displaystyle\int_2^\infty \frac{2}{v^2-v}\,dv$

12. $\displaystyle\int_2^\infty \frac{2\,dt}{t^2-1}$

13. $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$

14. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\displaystyle\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

18. $\displaystyle\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\displaystyle\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$

20. $\displaystyle\int_0^\infty \frac{16\tan^{-1} x}{1+x^2}\,dx$

21. $\displaystyle\int_{-\infty}^0 \theta e^\theta\,d\theta$

22. $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$

23. $\displaystyle\int_{-\infty}^0 e^{-|x|}\,dx$

24. $\displaystyle\int_{-\infty}^\infty 2xe^{-x^2}\,dx$

25. $\displaystyle\int_0^1 x\ln x\,dx$

26. $\displaystyle\int_0^1 (-\ln x)\,dx$

27. $\displaystyle\int_0^2 \frac{ds}{\sqrt{4-s^2}}$

28. $\displaystyle\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}}$

29. $\displaystyle\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$

30. $\displaystyle\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$

31. $\displaystyle\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

32. $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

33. $\displaystyle\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6}$

34. $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x^2+1)}$

1. $\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} [\tan^{-1} x]_0^b = \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

2. $\int_1^\infty \frac{dx}{x^{1.001}} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^{1.001}} = \lim_{b\to\infty} [-1000x^{-0.001}]_1^b = \lim_{b\to\infty} \left(\frac{-1000}{b^{0.001}} + 1000\right) = 1000$

3. $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 x^{-1/2}\,dx = \lim_{b\to 0^+} [2x^{1/2}]_b^1 = \lim_{b\to 0^+} \left(2 - 2\sqrt{b}\right) = 2 - 0 = 2$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta; \begin{bmatrix} u = \theta^2 + 2\theta \\ du = 2(\theta+1)\,d\theta \end{bmatrix} \rightarrow \int_0^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+} \int_b^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+} \left[\sqrt{u}\right]_b^3 = \lim_{b\to 0^+} \left(\sqrt{3} - \sqrt{b}\right)$
$= \sqrt{3} - 0 = \sqrt{3}$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds = \frac{1}{2}\int_0^2 \frac{2s\,ds}{\sqrt{4-s^2}} + \int_0^2 \frac{ds}{\sqrt{4-s^2}};\ \begin{bmatrix} u = 4 - s^2 \\ du = -2s\,ds \end{bmatrix} \rightarrow -\frac{1}{2}\int_4^0 \frac{du}{\sqrt{u}} + \lim_{c\to 2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}}$
$= \lim_{b\to 0^+} \int_b^4 \frac{du}{2\sqrt{u}} + \lim_{c\to 2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 0^+} \left[\sqrt{u}\right]_b^4 + \lim_{c\to 2^-} \left[\sin^{-1} \frac{s}{2}\right]_0^c$
$= \lim_{b\to 0^+} \left(2 - \sqrt{b}\right) + \lim_{c\to 2^-} \left(\sin^{-1} \frac{c}{2} - \sin^{-1} 0\right) = (2-0) + \left(\frac{\pi}{2} - 0\right) = \frac{4+\pi}{2}$

32. $\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^-} \left[-2\sqrt{1-x}\right]_0^b + \lim_{c\to 1^+} \left[2\sqrt{x-1}\right]_c^2$
$= \lim_{b\to 1^-} \left(-2\sqrt{1-b}\right) - \left(-2\sqrt{1-0}\right) + 2\sqrt{2-1} - \lim_{c\to 1^+} \left(2\sqrt{c-1}\right) = 0 + 2 + 2 - 0 = 4$

33. $\int_{-1}^\infty \frac{d\theta}{\theta^2 + 5\theta + 6} = \lim_{b\to\infty} \left[\ln\left|\frac{\theta+2}{\theta+3}\right|\right]_{-1}^b = \lim_{b\to\infty} \left[\ln\left|\frac{b+2}{b+3}\right|\right] - \ln\left|\frac{-1+2}{-1+3}\right| = 0 - \ln\left(\frac{1}{2}\right) = \ln 2$

CHAPTER FIVE

APPLICATIONS OF THE DEFINITE INTEGRAL

1- AREAS BETWEEN CURVES

DEFINITION Area Between Curves

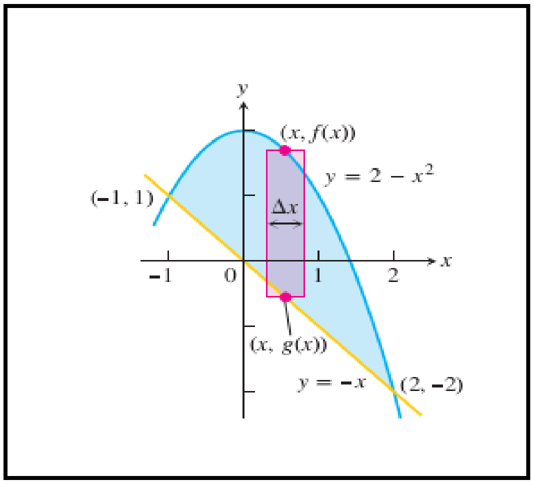
If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

EXAMPLE

Find the area of the region enclosed by the parabola and the line

$y = 2 - x^2$ and the line $y = -x$.



Solution

First we sketch the two curves.

Then find the limits of integration

$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rewrite.} \\ (x+1)(x-2) &= 0 && \text{Factor.} \\ x = -1, \quad x &= 2. && \text{Solve.} \end{aligned}$$

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx \\ &= \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} \\ &= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2} \end{aligned}$$

EXAMPLE

Find the area of the region in the first quadrant that is bounded above by **y= √x** and below by the x-axis and the line y= x – 2 .

Solution

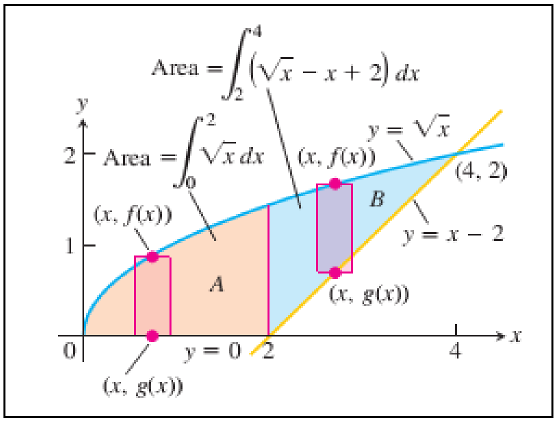
upper boundary is the graph of **f(x) = √x**

Lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$

$g(x) = x - 2$ for $2 \le x \le 4$

region A are $a = 0$ and $b = 2$ and B is $a = 2$, To find the right-hand limit

$\sqrt{x} = x - 2$	Equate $f(x)$ and $g(x)$.
$x = (x-2)^2 = x^2 - 4x + 4$	Square both sides.
$x^2 - 5x + 4 = 0$	Rewrite.
$(x-1)(x-4) = 0$	Factor.
$x = 1, \quad x = 4.$	Solve.



For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x-2) = \sqrt{x} - x + 2$

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

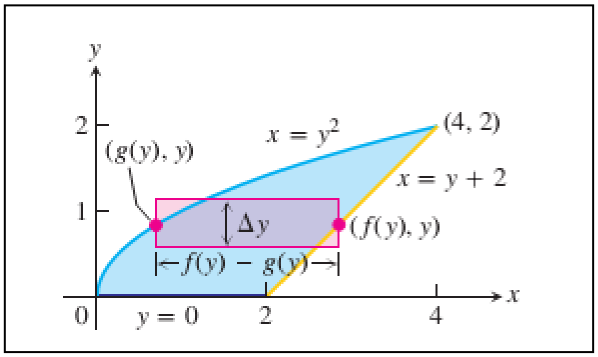
$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

EXAMPLE

Find the area of the region in the first quadrant that is bounded above by **y= √x** and below by the x-axis and the line y= x – 2 . with respect to y.

$$y + 2 = y^2 \quad \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2.$$
$$y^2 - y - 2 = 0 \quad \text{Rewrite.}$$
$$(y+1)(y-2) = 0 \quad \text{Factor.}$$
$$y = -1, \quad y = 2 \quad \text{Solve.}$$

The upper limit of integration is b=2 (The value y= -1 gives a point of intersection below the x-axis.) The area of the region is.



$$A = \int_a^b [f(y) - g(y)]\, dy = \int_0^2 [y + 2 - y^2]\, dy$$

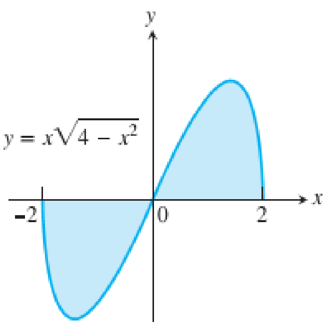
$$= \int_0^2 [2 + y - y^2]\, dy$$
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$
$$= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}.$$

EXERCISES 5.6

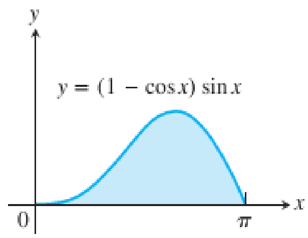
Area

Find the total areas of the shaded regions in Exercises 25–40.

25.



26.



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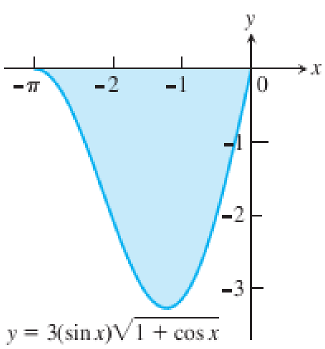
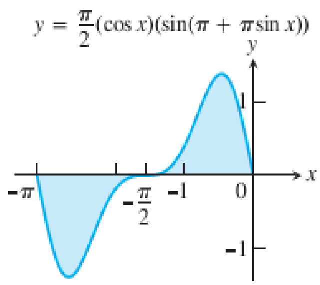
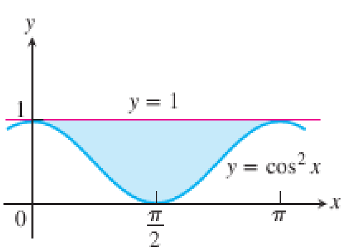


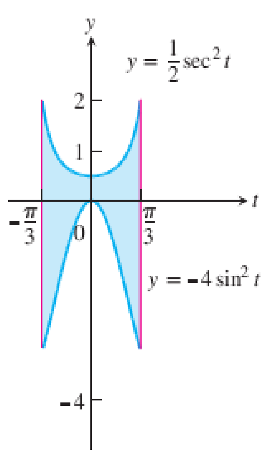
28.



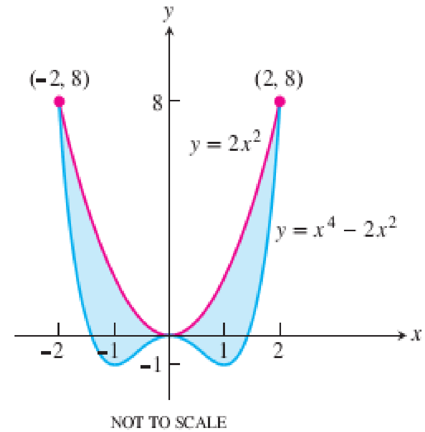
29.



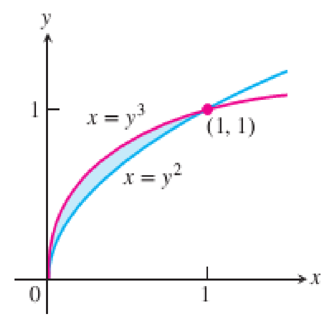
30.



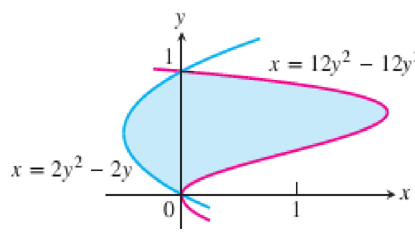
31.



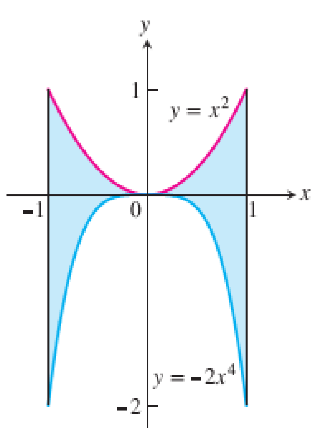
32.



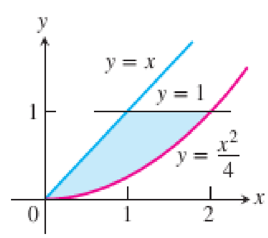
33.

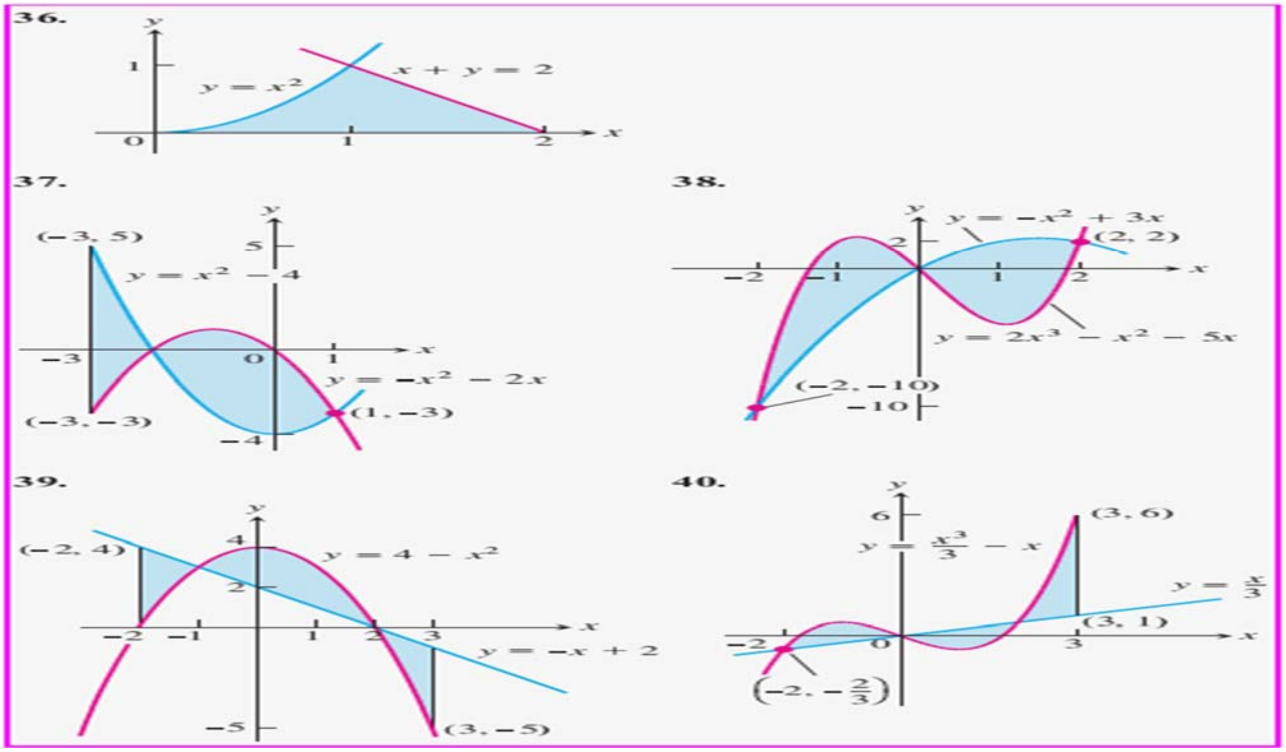


34.



35.





25. Let $u = 4 - x^2 \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}\,du = x\,dx$; $x = -2 \Rightarrow u = 0$, $x = 0 \Rightarrow u = 4$, $x = 2 \Rightarrow u = 0$

$$A = -\int_{-2}^{0} x\sqrt{4 - x^2}\,dx + \int_{0}^{2} x\sqrt{4 - x^2}\,dx = -\int_{0}^{4} -\frac{1}{2}u^{1/2}\,du + \int_{4}^{0} -\frac{1}{2}u^{1/2}\,du = 2\int_{0}^{4}\frac{1}{2}u^{1/2}\,du = \int_{0}^{4} u^{1/2}\,du$$

$$= \left[\frac{2}{3}u^{3/2}\right]_0^4 = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{16}{3}$$

26. Let $u = 1 - \cos x \Rightarrow du = \sin x\,dx$; $x = 0 \Rightarrow u = 0$, $x = \pi \Rightarrow u = 2$

$$\int_0^{\pi} (1 - \cos x)\sin x\,dx = \int_0^2 u\,du = \left[\frac{u^2}{2}\right]_0^2 = \frac{2^2}{2} - \frac{0^2}{2} = 2$$

27. Let $u = 1 + \cos x \Rightarrow du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$; $x = -\pi \Rightarrow u = 1 + \cos(-\pi) = 0$, $x = 0$ $\Rightarrow u = 1 + \cos 0 = 2$

$$A = -\int_{-\pi}^{0} 3(\sin x)\sqrt{1 + \cos x}\,dx = -\int_0^2 3u^{1/2}(-du) = 3\int_0^2 u^{1/2}\,du = \left[2u^{3/2}\right]_0^2 = 2(2)^{3/2} - 2(0)^{3/2} = 2^{5/2}$$

33. For the sketch given, $c = 0$, $d = 1$; $f(y) - g(y) = (12y^2 - 12y^3) - (2y^2 - 2y) = 10y^2 - 12y^3 + 2y$;

$$A = \int_0^1 (10y^2 - 12y^3 + 2y)\,dy = \int_0^1 10y^2\,dy - \int_0^1 12y^3\,dy + \int_0^1 2y\,dy = \left[\tfrac{10}{3}y^3\right]_0^1 - \left[\tfrac{12}{4}y^4\right]_0^1 + \left[\tfrac{2}{2}y^2\right]_0^1$$

$$= \left(\tfrac{10}{3} - 0\right) - (3 - 0) + (1 - 0) = \tfrac{4}{3}$$

34. For the sketch given, $a = -1$, $b = 1$; $f(x) - g(x) = x^2 - (-2x^4) = x^2 + 2x^4$;

$$A = \int_{-1}^1 (x^2 + 2x^4)\,dx = \left[\tfrac{x^3}{3} + \tfrac{2x^5}{5}\right]_{-1}^1 = \left(\tfrac{1}{3} + \tfrac{2}{5}\right) - \left[-\tfrac{1}{3} + \left(-\tfrac{2}{5}\right)\right] = \tfrac{2}{3} + \tfrac{4}{5} = \tfrac{10+12}{15} = \tfrac{22}{15}$$

39. AREA $=$ A1 $+$ A2 $+$ A3

A1: For the sketch given, $a = -2$ and $b = -1$: $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$

$$\Rightarrow A1 = \int_{-2}^{-1} (x^2 - x - 2)\,dx = \left[\tfrac{x^3}{3} - \tfrac{x^2}{2} - 2x\right]_{-2}^{-1} = \left(-\tfrac{1}{3} - \tfrac{1}{2} + 2\right) - \left(-\tfrac{8}{3} - \tfrac{4}{2} + 4\right) = \tfrac{7}{3} - \tfrac{1}{2} = \tfrac{14-3}{6} = \tfrac{11}{6};$$

A2: For the sketch given, $a = -1$ and $b = 2$: $f(x) - g(x) = (4 - x^2) - (-x + 2) = -(x^2 - x - 2)$

$$\Rightarrow A2 = -\int_{-1}^{2} (x^2 - x - 2)\,dx = -\left[\tfrac{x^3}{3} - \tfrac{x^2}{2} - 2x\right]_{-1}^{2} = -\left(\tfrac{8}{3} - \tfrac{4}{2} - 4\right) + \left(-\tfrac{1}{3} - \tfrac{1}{2} + 2\right) = -3 + 8 - \tfrac{1}{2} = \tfrac{9}{2};$$

A3: For the sketch given, $a = 2$ and $b = 3$: $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$

$$\Rightarrow A3 = \int_{2}^{3} (x^2 - x - 2)\,dx = \left[\tfrac{x^3}{3} - \tfrac{x^2}{2} - 2x\right]_{2}^{3} = \left(\tfrac{27}{3} - \tfrac{9}{2} - 6\right) - \left(\tfrac{8}{3} - \tfrac{4}{2} - 4\right) = 9 - \tfrac{9}{2} - \tfrac{8}{3};$$

Therefore, AREA $=$ A1 $+$ A2 $+$ A3 $= \tfrac{11}{6} + \tfrac{9}{2} + \left(9 - \tfrac{9}{2} - \tfrac{8}{3}\right) = 9 - \tfrac{5}{6} = \tfrac{49}{6}$

40. AREA $=$ A1 $+$ A2 $+$ A3

A1: For the sketch given, $a = -2$ and $b = 0$: $f(x) - g(x) = \left(\tfrac{x^3}{3} - x\right) - \tfrac{x}{3} = \tfrac{x^3}{3} - \tfrac{4}{3}x = \tfrac{1}{3}(x^3 - 4x)$

$$\Rightarrow A1 = \tfrac{1}{3}\int_{-2}^{0} (x^3 - 4x)\,dx = \tfrac{1}{3}\left[\tfrac{x^4}{4} - 2x^2\right]_{-2}^{0} = 0 - \tfrac{1}{3}(4 - 8) = \tfrac{4}{3};$$

A2: For the sketch given, $a = 0$ and we find b by solving the equations $y = \tfrac{x^3}{3} - x$ and $y = \tfrac{x}{3}$ simultaneously for x: $\tfrac{x^3}{3} - x = \tfrac{x}{3} \Rightarrow \tfrac{x^3}{3} - \tfrac{4}{3}x = 0 \Rightarrow \tfrac{x}{3}(x - 2)(x + 2) = 0 \Rightarrow x = -2, x = 0$, or $x = 2$ so $b = 2$:

$$f(x) - g(x) = \tfrac{x}{3} - \left(\tfrac{x^3}{3} - x\right) = -\tfrac{1}{3}(x^3 - 4x) \Rightarrow A2 = -\tfrac{1}{3}\int_0^2 (x^3 - 4x)\,dx = \tfrac{1}{3}\int_0^2 (4x - x^3) = \tfrac{1}{3}\left[2x^2 - \tfrac{x^4}{4}\right]_0^2$$

$$= \tfrac{1}{3}(8 - 4) = \tfrac{4}{3};$$

A3: For the sketch given, $a = 2$ and $b = 3$: $f(x) - g(x) = \left(\tfrac{x^3}{3} - x\right) - \tfrac{x}{3} = \tfrac{1}{3}(x^3 - 4x)$

$$\Rightarrow A3 = \tfrac{1}{3}\int_2^3 (x^3 - 4x)\,dx = \tfrac{1}{3}\left[\tfrac{x^4}{4} - 2x^2\right]_2^3 = \tfrac{1}{3}\left[\left(\tfrac{81}{4} - 2 \cdot 9\right) - \left(\tfrac{16}{4} - 8\right)\right] = \tfrac{1}{3}\left(\tfrac{81}{4} - 14\right) = \tfrac{25}{12};$$

Therefore, AREA $=$ A1 $+$ A2 $+$ A3 $= \tfrac{4}{3} + \tfrac{4}{3} + \tfrac{25}{12} = \tfrac{32+25}{12} = \tfrac{19}{4}$

Find the areas of the regions enclosed by the lines and curves in Exercises 41–50.

41. $y = x^2 - 2$ and $y = 2$

42. $y = 2x - x^2$ and $y = -3$

43. $y = x^4$ and $y = 8x$

44. $y = x^2 - 2x$ and $y = x$

45. $y = x^2$ and $y = -x^2 + 4x$

46. $y = 7 - 2x^2$ and $y = x^2 + 4$

47. $y = x^4 - 4x^2 + 4$ and $y = x^2$

48. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

49. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

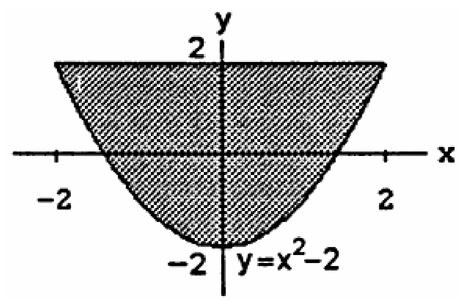
50. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

41. $a = -2, b = 2;$

$f(x) - g(x) = 2 - (x^2 - 2) = 4 - x^2$

$$\Rightarrow A = \int_{-2}^{2} (4 - x^2)dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= 2 \cdot \left(\frac{24}{3} - \frac{8}{3}\right) = \frac{32}{3}$$



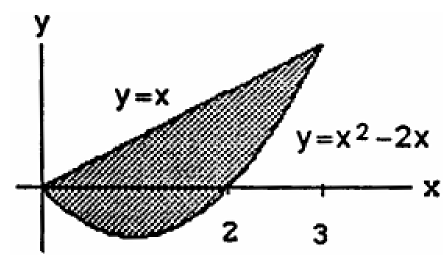
44. Limits of integration: $x^2 - 2x = x \Rightarrow x^2 = 3x$

$\Rightarrow x(x - 3) = 0 \Rightarrow a = 0$ and $b = 3;$

$f(x) - g(x) = x - (x^2 - 2x) = 3x - x^2$

$$\Rightarrow A = \int_{0}^{3} (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_{0}^{3}$$

$$= \frac{27}{2} - 9 = \frac{27 - 18}{2} = \frac{9}{2}$$



47. Limits of integration: $x^4 - 4x^2 + 4 = x^2$

$\Rightarrow x^4 - 5x^2 + 4 = 0 \Rightarrow (x^2 - 4)(x^2 - 1) = 0$

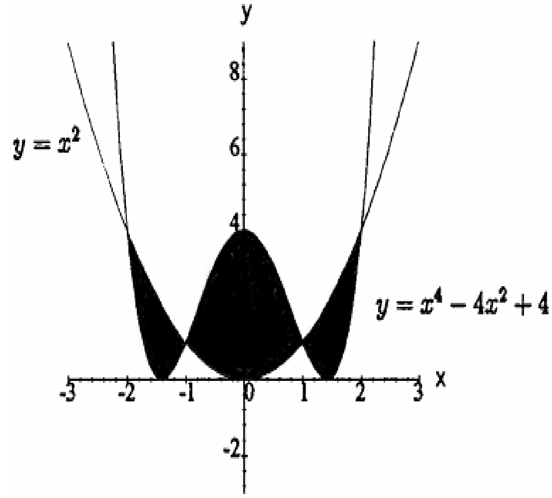
$\Rightarrow (x+2)(x-2)(x+1)(x-1) = 0 \Rightarrow x = -2, -1, 1, 2;$

$f(x) - g(x) = (x^4 - 4x^2 + 4) - x^2 = x^4 - 5x^2 + 4$ and

$g(x) - f(x) = x^2 - (x^4 - 4x^2 + 4) = -x^4 + 5x^2 - 4$

$\Rightarrow A = \int_{-2}^{-1}(-x^4 + 5x^2 - 4)dx + \int_{-1}^{1}(x^4 - 5x^2 + 4)dx$

$+ \int_{1}^{2}(-x^4 + 5x^2 - 4)dx$



$= \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_{-2}^{-1} + \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_{-1}^{1} + \left[\frac{-x^5}{5} + \frac{5x^3}{3} - 4x\right]_{1}^{2}$

$= \left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(\frac{32}{5} - \frac{40}{3} + 8\right) + \left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right) + \left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)$

$= -\frac{60}{5} + \frac{60}{3} = \frac{300-180}{15} = 8$

Find the areas of the regions enclosed by the lines and curves in Exercises 51–58.

51. $x = 2y^2, \quad x = 0, \quad$ and $\quad y = 3$
52. $x = y^2 \quad$ and $\quad x = y + 2$
53. $y^2 - 4x = 4 \quad$ and $\quad 4x - y = 16$
54. $x - y^2 = 0 \quad$ and $\quad x + 2y^2 = 3$
55. $x + y^2 = 0 \quad$ and $\quad x + 3y^2 = 2$
56. $x - y^{2/3} = 0 \quad$ and $\quad x + y^4 = 2$
57. $x = y^2 - 1 \quad$ and $\quad x = |y|\sqrt{1 - y^2}$
58. $x = y^3 - y^2 \quad$ and $\quad x = 2y$

54. Limits of integration: $x = y^2$ and $x = 3 - 2y^2$

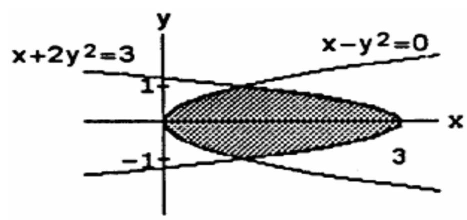
$\Rightarrow y^2 = 3 - 2y^2 \Rightarrow 3y^2 = 3 \Rightarrow 3(y-1)(y+1) = 0$

$\Rightarrow c = -1$ and $d = 1$; $f(y) - g(y) = (3 - 2y^2) - y^2$

$= 3 - 3y^2 = 3(1 - y^2) \Rightarrow A = 3\int_{-1}^{1}(1 - y^2)\,dy$

$= 3\left[y - \frac{y^3}{3}\right]_{-1}^{1} = 3\left(1 - \frac{1}{3}\right) - 3\left(-1 + \frac{1}{3}\right)$

$= 3 \cdot 2\left(1 - \frac{1}{3}\right) = 4$



55. Limits of integration: $x = -y^2$ and $x = 2 - 3y^2$

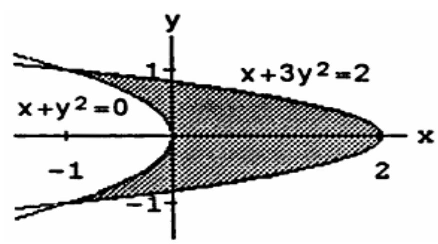
$\Rightarrow -y^2 = 2 - 3y^2 \Rightarrow 2y^2 - 2 = 0$

$\Rightarrow 2(y-1)(y+1) = 0 \Rightarrow c = -1$ and $d = 1$;

$f(y) - g(y) = (2 - 3y^2) - (-y^2) = 2 - 2y^2 = 2(1 - y^2)$

$\Rightarrow A = 2\int_{-1}^{1}(1 - y^2)\,dy = 2\left[y - \frac{y^3}{3}\right]_{-1}^{1}$

$= 2\left(1 - \frac{1}{3}\right) - 2\left(-1 + \frac{1}{3}\right) = 4\left(\frac{2}{3}\right) = \frac{8}{3}$



Find the areas of the regions enclosed by the curves in Exercises 59–62.

59. $4x^2 + y = 4$ and $x^4 - y = 1$

60. $x^3 - y = 0$ and $3x^2 - y = 4$

61. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

62. $x + y^2 = 3$ and $4x + y^2 = 0$

59. Limits of integration: $y = -4x^2 + 4$ and $y = x^4 - 1$

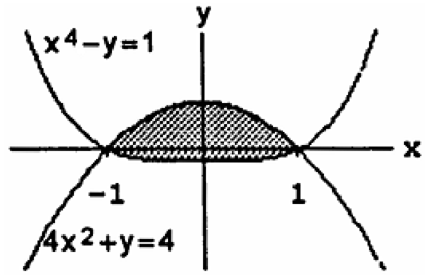
$$\Rightarrow x^4 - 1 = -4x^2 + 4 \Rightarrow x^4 + 4x^2 - 5 = 0$$

$$\Rightarrow (x^2 + 5)(x - 1)(x + 1) = 0 \Rightarrow a = -1 \text{ and } b = 1;$$

$$f(x) - g(x) = -4x^2 + 4 - x^4 + 1 = -4x^2 - x^4 + 5$$

$$\Rightarrow A = \int_{-1}^{1} (-4x^2 - x^4 + 5)\, dx = \left[-\tfrac{4x^3}{3} - \tfrac{x^5}{5} + 5x\right]_{-1}^{1}$$

$$= \left(-\tfrac{4}{3} - \tfrac{1}{5} + 5\right) - \left(\tfrac{4}{3} + \tfrac{1}{5} - 5\right) = 2\left(-\tfrac{4}{3} - \tfrac{1}{5} + 5\right) = \tfrac{104}{15}$$



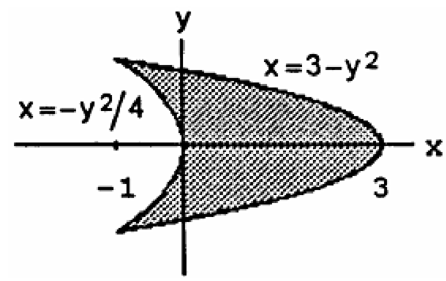
62. Limits of integration: $x = 3 - y^2$ and $x = -\frac{y^2}{4}$

$$\Rightarrow 3 - y^2 = -\tfrac{y^2}{4} \Rightarrow \tfrac{3y^2}{4} - 3 = 0 \Rightarrow \tfrac{3}{4}(y - 2)(y + 2) = 0$$

$$\Rightarrow c = -2 \text{ and } d = 2;\ f(y) - g(y) = (3 - y^2) - \left(\tfrac{-y^2}{4}\right)$$

$$= 3\left(1 - \tfrac{y^2}{4}\right) \Rightarrow A = 3\int_{-2}^{2} \left(1 - \tfrac{y^2}{4}\right) dy = 3\left[y - \tfrac{y^3}{12}\right]_{-2}^{2}$$

$$= 3\left[\left(2 - \tfrac{8}{12}\right) - \left(-2 + \tfrac{8}{12}\right)\right] = 3\left(4 - \tfrac{16}{12}\right) = 12 - 4 = 8$$



Find the areas of the regions enclosed by the lines and curves in Exercises 63–70.

63. $y = 2\sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

64. $y = 8\cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

65. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

66. $y = \sin(\pi x/2)$ and $y = x$

67. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

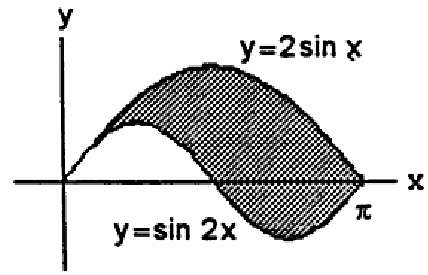
68. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

69. $x = 3\sin y\sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

70. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

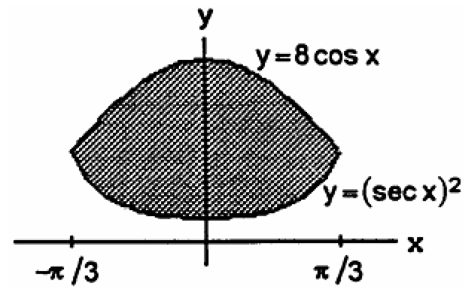
63. $a = 0, b = \pi$; $f(x) - g(x) = 2\sin x - \sin 2x$

$$\Rightarrow A = \int_0^{\pi} (2\sin x - \sin 2x)\,dx = \left[-2\cos x + \frac{\cos 2x}{2}\right]_0^{\pi}$$
$$= \left[-2(-1) + \tfrac{1}{2}\right] - \left(-2 \cdot 1 + \tfrac{1}{2}\right) = 4$$



64. $a = -\frac{\pi}{3}, b = \frac{\pi}{3}$; $f(x) - g(x) = 8\cos x - \sec^2 x$

$$\Rightarrow A = \int_{-\pi/3}^{\pi/3} (8\cos x - \sec^2 x)\,dx = [8\sin x - \tan x]_{-\pi/3}^{\pi/3}$$
$$= \left(8 \cdot \tfrac{\sqrt{3}}{2} - \sqrt{3}\right) - \left(-8 \cdot \tfrac{\sqrt{3}}{2} + \sqrt{3}\right) = 6\sqrt{3}$$



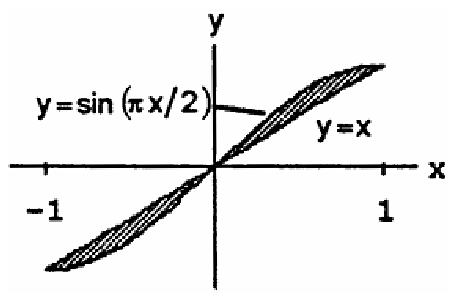
66. $A = A1 + A2$

$a_1 = -1, b_1 = 0$ and $a_2 = 0, b_2 = 1$;

$f_1(x) - g_1(x) = x - \sin\left(\frac{\pi x}{2}\right)$ and $f_2(x) - g_2(x) = \sin\left(\frac{\pi x}{2}\right) - x$

$\Rightarrow$ by symmetry about the origin,

$$A_1 + A_2 = 2A_1 \Rightarrow A = 2\int_0^1 \left[\sin\left(\tfrac{\pi x}{2}\right) - x\right] dx$$
$$= 2\left[-\tfrac{2}{\pi}\cos\left(\tfrac{\pi x}{2}\right) - \tfrac{x^2}{2}\right]_0^1 = 2\left[\left(-\tfrac{2}{\pi} \cdot 0 - \tfrac{1}{2}\right) - \left(-\tfrac{2}{\pi} \cdot 1 - 0\right)\right]$$
$$= 2\left(\tfrac{2}{\pi} - \tfrac{1}{2}\right) = 2\left(\tfrac{4-\pi}{2\pi}\right) = \tfrac{4-\pi}{\pi}$$



71. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

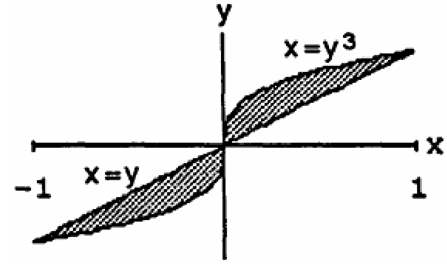
72. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

73. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

74. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

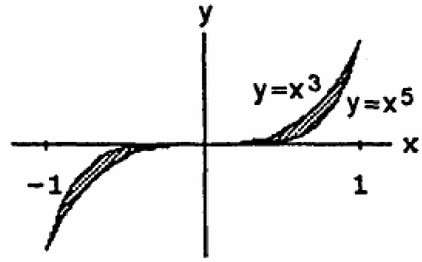
71. $A = A_1 + A_2$

Limits of integration: $x = y^3$ and $x = y \Rightarrow y = y^3$
$\Rightarrow y^3 - y = 0 \Rightarrow y(y-1)(y+1) = 0 \Rightarrow c_1 = -1, d_1 = 0$
and $c_2 = 0, d_2 = 1$; $f_1(y) - g_1(y) = y^3 - y$ and
$f_2(y) - g_2(y) = y - y^3 \Rightarrow$ by symmetry about the origin,
$A_1 + A_2 = 2A_2 \Rightarrow A = 2\int_0^1 (y - y^3)\, dy = 2\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$
$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$



72. $A = A_1 + A_2$

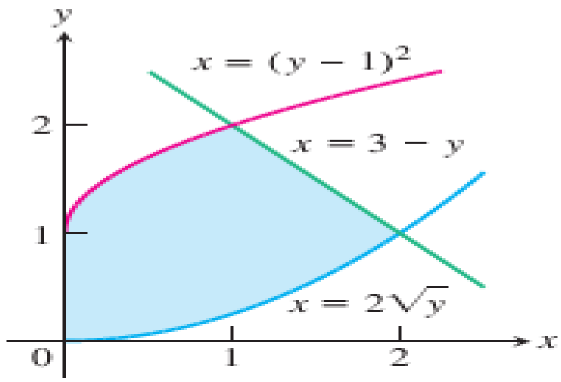
Limits of integration: $y = x^3$ and $y = x^5 \Rightarrow x^3 = x^5$
$\Rightarrow x^5 - x^3 = 0 \Rightarrow x^3(x-1)(x+1) = 0 \Rightarrow a_1 = -1, b_1 = 0$
and $a_2 = 0, b_2 = 1$; $f_1(x) - g_1(x) = x^3 - x^5$ and
$f_2(x) - g_2(x) = x^5 - x^3 \Rightarrow$ by symmetry about the origin,
$A_1 + A_2 = 2A_2 \Rightarrow A = 2\int_0^1 (x^3 - x^5)\, dx = 2\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$
$= 2\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{6}$



76. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** x, **b.** y.

77. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

78. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y-1)^2$, and above right by the line $x = 3 - y$.



2- VOLUMES

DEFINITION Volume

The **volume** of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

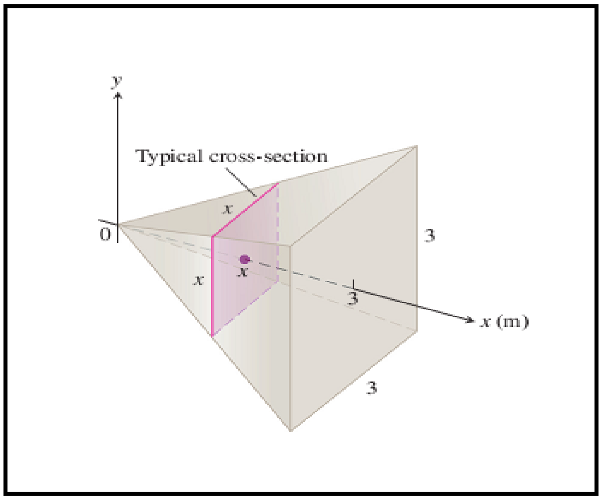
❖ To apply the formula in the definition to calculate the volume of a solid, take the following steps:

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for $A(x)$,* the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate $A(x)$* using the Fundamental Theorem.

EXAMPLE

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.



1. A sketch the pyramid with its altitude along the x-axis and its vertex at the origin and include a typical cross-section.

2. A formula for A(x). The cross-section at x is a square x meters on a side, so its area is $A(x) = x^2.$

3. The limits of integration. The squares lie on the planes from $x = 0$ to $x = 3.$

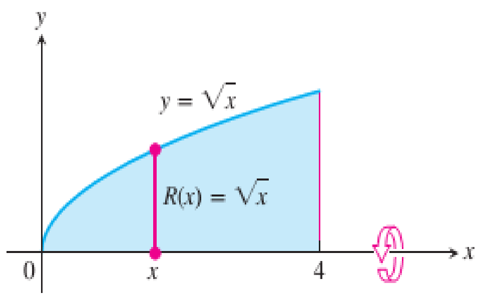
4. Integrate to find the volume

$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big]_0^3 = 9\text{ m}^3$$

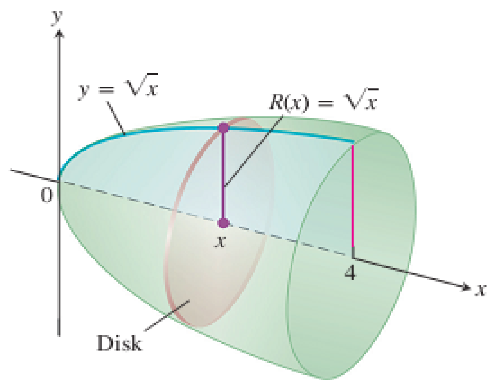
Solids of Revolution: The Disk Method

- The solid generated by rotating a plane region about an axis in its plane is called a solid of revolution.

- To find the volume of a solid like the one shown below ,
- We need only observe that the cross-sectional area A(x) is the area of a disk of radius R(x), the distance of the planar region's boundary from the axis of revolution.



(a)



(b)

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume gives

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

EXAMPLE

The region between the curve $y = \sqrt{x}, 0 \le x \le 4$ and the x-axis is revolved about the x-axis to generate a solid. Find its volume

Solution The volume is

$$\begin{aligned} V &= \int_a^b \pi[R(x)]^2\,dx \\ &= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx \qquad R(x) = \sqrt{x} \\ &= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\Big]_0^4 = \pi\frac{(4)^2}{2} = 8\pi. \end{aligned}$$

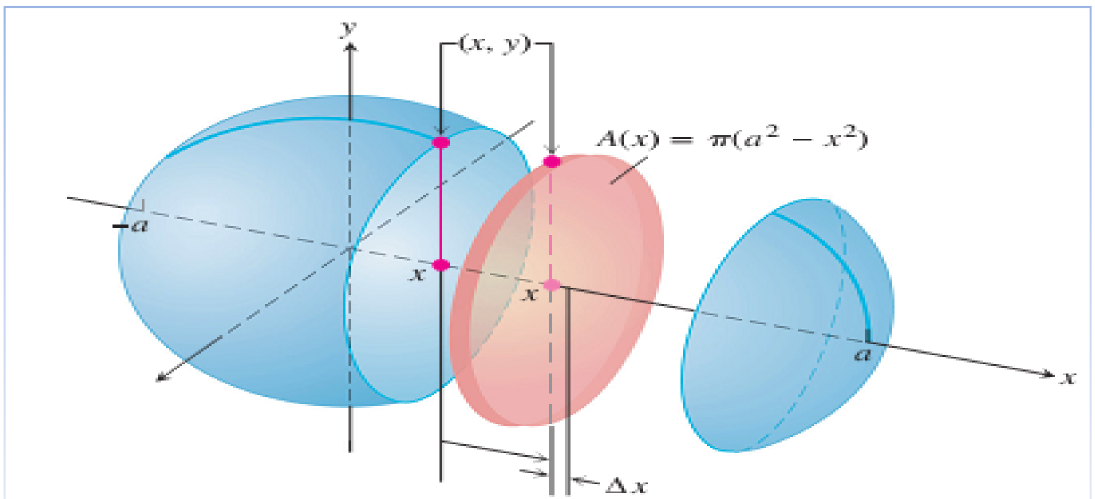
EXAMPLE

The circle $x^2 + y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find its volume.

Solution we imagine the sphere cut into thin slices by planes perpendicular to the x-axis .The cross-sectional area at a typical point x between -a and a is

$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$



EXAMPLE A Solid of Revolution (Rotation About the Line)

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1, x = 4$ about the line y=1 .

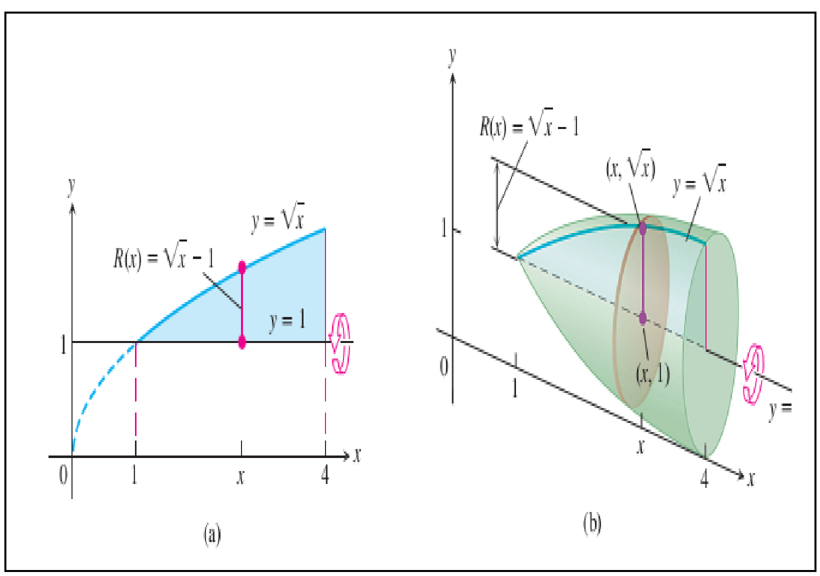
Solution

$$V = \int_1^4 \pi[R(x)]^2\,dx$$

$$= \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx$$

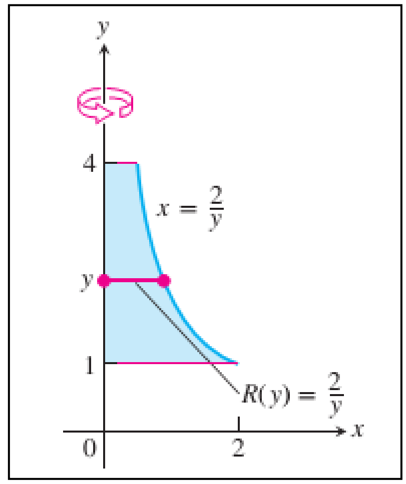
$$= \pi\int_1^4 \left[x - 2\sqrt{x} + 1\right] dx$$

$$= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}.$$



EXAMPLE 7 Rotation About the y-Axis

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x=2/y$, $1\leq y\leq 4$ about the y-axis.



Solution

$$V = \int_1^4 \pi[R(y)]^2\, dy$$
$$= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy$$
$$= \pi\int_1^4 \frac{4}{y^2}\, dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right]$$
$$= 3\pi.$$

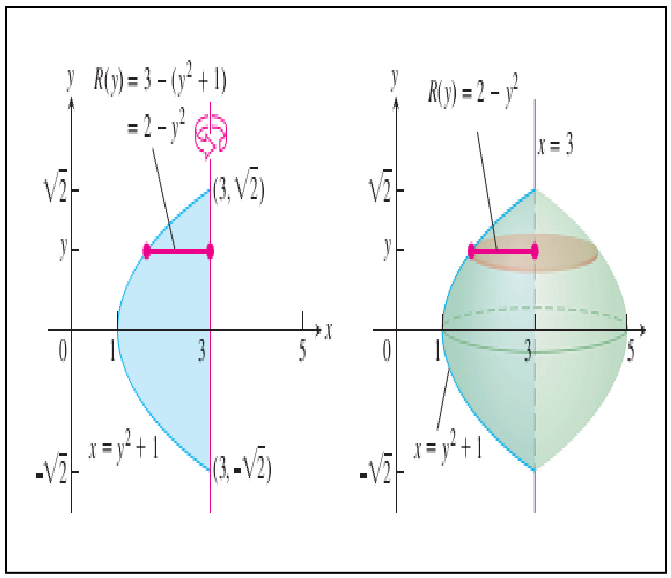
EXAMPLE 8 Rotation About a Vertical Axis

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.12). Note that the cross-sections are perpendicular to the line $x = 3$. The volume is

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2\, dy$$
$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2\, dy$$
$$= \pi\int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\, dy$$
$$= \pi\left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}}$$
$$= \frac{64\pi\sqrt{2}}{15}.$$

$R(y) = 3 - (y^2 + 1)$
$= 2 - y^2$



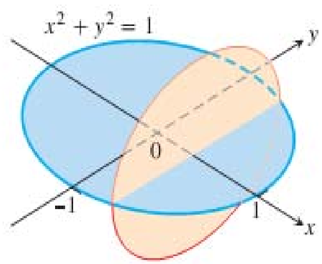
EXERCISES 6.1

Cross-Sectional Areas

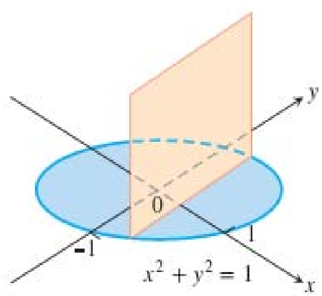
In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross-sections of the solid perpendicular to the x-axis.

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. In each case, the cross-sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

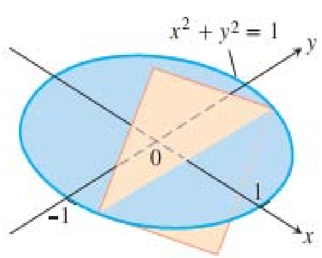
a. The cross-sections are circular disks with diameters in the xy-plane.



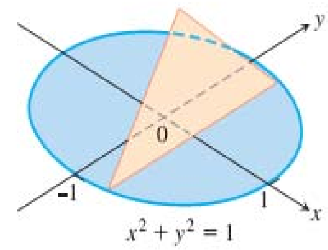
b. The cross-sections are squares with bases in the xy-plane.



c. The cross-sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)

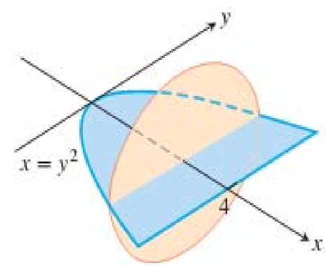


d. The cross-sections are equilateral triangles with bases in the xy-plane.

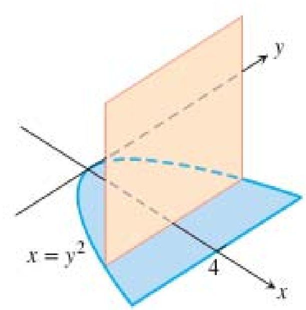


2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the x-axis between these planes run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

a. The cross-sections are circular disks with diameters in the xy-plane.



b. The cross-sections are squares with bases in the xy-plane.



c. The cross-sections are squares with diagonals in the xy-plane.

d. The cross-sections are equilateral triangles with bases in the xy-plane.

1. (a) $A = \pi(\text{radius})^2$ and radius $= \sqrt{1-x^2} \Rightarrow A(x) = \pi\left(1-x^2\right)$

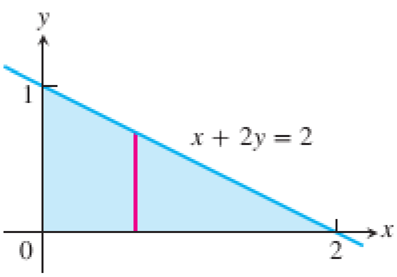
(b) $A = \text{width} \cdot \text{height}$, width $=$ height $= 2\sqrt{1-x^2} \Rightarrow A(x) = 4\left(1-x^2\right)$

(c) $A = (\text{side})^2$ and diagonal $= \sqrt{2}(\text{side}) \Rightarrow A = \frac{(\text{diagonal})^2}{2}$; diagonal $= 2\sqrt{1-x^2} \Rightarrow A(x) = 2\left(1-x^2\right)$

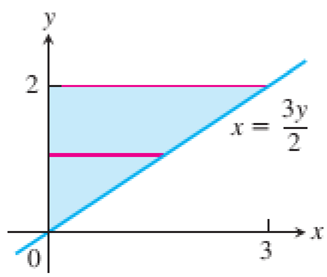
(d) $A = \frac{\sqrt{3}}{4}(\text{side})^2$ and side $= 2\sqrt{1-x^2} \Rightarrow A(x) = \sqrt{3}\left(1-x^2\right)$

In Exercises 13–16, find the volume of the solid generated by revolving the shaded region about the given axis.

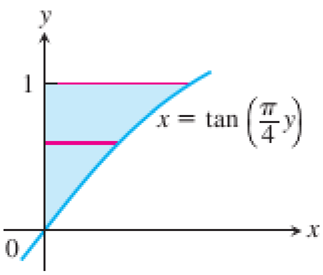
13. About the x-axis



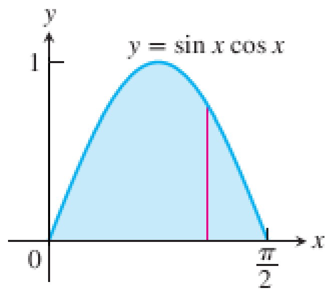
14. About the y-axis



15. About the y-axis



16. About the x-axis



13. $R(x) = y = 1 - \frac{x}{2} \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 \left(1-\frac{x}{2}\right)^2 dx = \pi\int_0^2 \left(1 - x + \frac{x^2}{4}\right) dx = \pi\left[x - \frac{x^2}{2} + \frac{x^3}{12}\right]_0^2$

$= \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$

15. $R(x) = \tan\left(\frac{\pi}{4}y\right)$; $u = \frac{\pi}{4}y \Rightarrow du = \frac{\pi}{4}dy \Rightarrow 4\,du = \pi\,dy$; $y = 0 \Rightarrow u = 0$, $y = 1 \Rightarrow u = \frac{\pi}{4}$;

$V = \int_0^1 \pi[R(y)]^2\,dy = \pi\int_0^1 \left[\tan\left(\frac{\pi}{4}y\right)\right]^2 dy = 4\int_0^{\pi/4} \tan^2 u\,du = 4\int_0^{\pi/4} \left(-1 + \sec^2 u\right) du = 4[-u + \tan u]_0^{\pi/4}$

$= 4\left(-\frac{\pi}{4} + 1 - 0\right) = 4 - \pi$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 17–22 about the x-axis.

17. $y = x^2, \quad y = 0, \quad x = 2$ **18.** $y = x^3, \quad y = 0, \quad x = 2$

19. $y = \sqrt{9 - x^2}, \quad y = 0$ **20.** $y = x - x^2, \quad y = 0$

21. $y = \sqrt{\cos x}, \quad 0 \le x \le \pi/2, \quad y = 0, \quad x = 0$

22. $y = \sec x, \quad y = 0, \quad x = -\pi/4, \quad x = \pi/4$

In Exercises 23 and 24, find the volume of the solid generated by revolving the region about the given line.

23. The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$

24. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 25–30 about the y-axis.

25. The region enclosed by $x = \sqrt{5}y^2, \quad x = 0, \quad y = -1, \quad y = 1$

26. The region enclosed by $x = y^{3/2}, \quad x = 0, \quad y = 2$

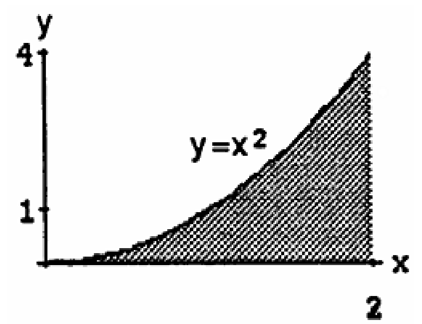
27. The region enclosed by $x = \sqrt{2 \sin 2y}, \quad 0 \le y \le \pi/2, \quad x = 0$

28. The region enclosed by $x = \sqrt{\cos(\pi y/4)}, \quad -2 \le y \le 0, \quad x = 0$

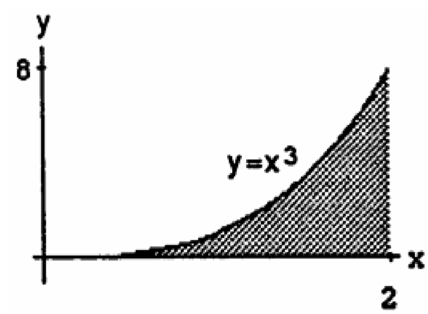
29. $x = 2/(y + 1), \quad x = 0, \quad y = 0, \quad y = 3$

30. $x = \sqrt{2y}/(y^2 + 1), \quad x = 0, \quad y = 1$

17. $R(x) = x^2 \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi \int_0^2 (x^2)^2\,dx$
$$= \pi \int_0^2 x^4\,dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$$



18. $R(x) = x^3 \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 (x^3)^2\,dx$
$$= \pi \int_0^2 x^6\,dx = \pi \left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$$



23. $R(x) = \sqrt{2} - \sec x \tan x \Rightarrow V = \int_0^{\pi/4} \pi[R(x)]^2\, dx$

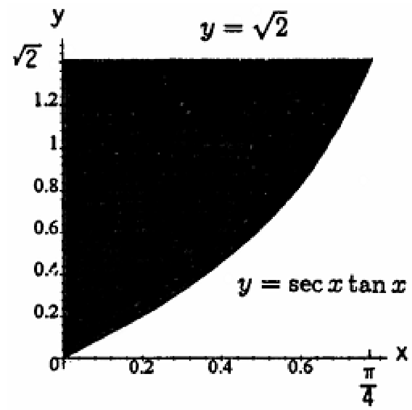
$$= \pi \int_0^{\pi/4} \left(\sqrt{2} - \sec x \tan x\right)^2 dx$$

$$= \pi \int_0^{\pi/4} \left(2 - 2\sqrt{2}\sec x \tan x + \sec^2 x \tan^2 x\right) dx$$

$$= \pi \left(\int_0^{\pi/4} 2\, dx - 2\sqrt{2}\int_0^{\pi/4} \sec x \tan x\, dx + \int_0^{\pi/4} (\tan x)^2 \sec^2 x\, dx\right)$$

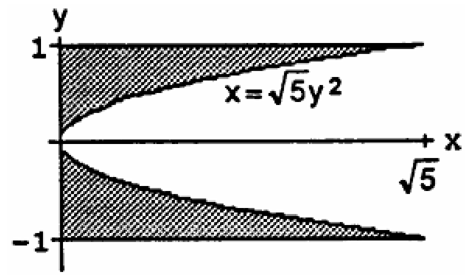
$$= \pi \left([2x]_0^{\pi/4} - 2\sqrt{2}\,[\sec x]_0^{\pi/4} + \left[\frac{\tan^3 x}{3}\right]_0^{\pi/4}\right)$$

$$= \pi \left[\left(\frac{\pi}{2} - 0\right) - 2\sqrt{2}\left(\sqrt{2} - 1\right) + \frac{1}{3}\left(1^3 - 0\right)\right] = \pi\left(\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3}\right)$$



25. $R(y) = \sqrt{5} \cdot y^2 \Rightarrow V = \int_{-1}^{1} \pi[R(y)]^2\, dy = \pi \int_{-1}^{1} 5y^4\, dy$

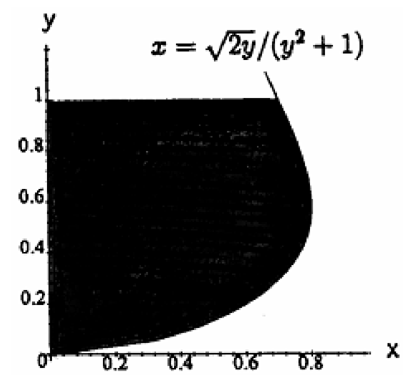
$$= \pi\,[y^5]_{-1}^{1} = \pi[1 - (-1)] = 2\pi$$



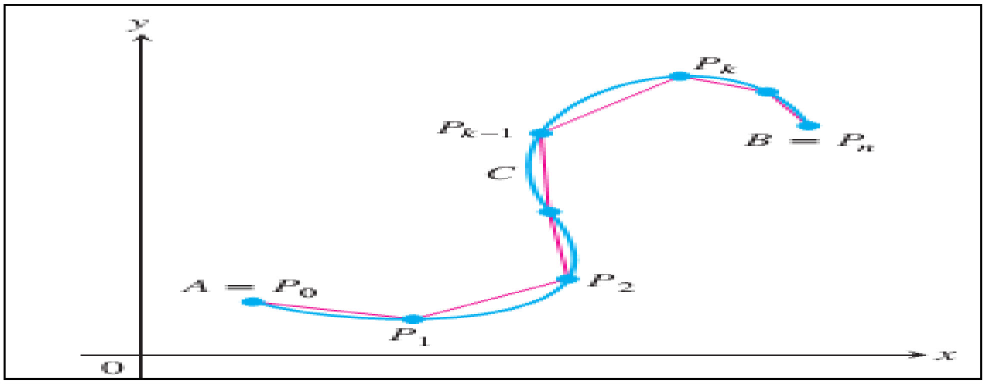
30. $R(y) = \frac{\sqrt{2y}}{y^2+1} \Rightarrow V = \int_0^1 \pi[R(y)]^2\, dy = \pi\int_0^1 2y\,(y^2+1)^{-2}\, dy;$

$[u = y^2 + 1 \Rightarrow du = 2y\, dy;\ y = 0 \Rightarrow u = 1,\ y = 1 \Rightarrow u = 2]$

$$\rightarrow V = \pi\int_1^2 u^{-2}\, du = \pi\left[-\frac{1}{u}\right]_1^2 = \pi\left[-\frac{1}{2} - (-1)\right] = \frac{\pi}{2}$$



3- LENGTHS OF PLANE CURVES



$x = f(t)$ and $y = g(t)$, $a \le t \le b$.

$A = (f(a), g(a))$ at time $t = a$,,,, $B = (f(b), g(b))$

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$
$$= \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}$$

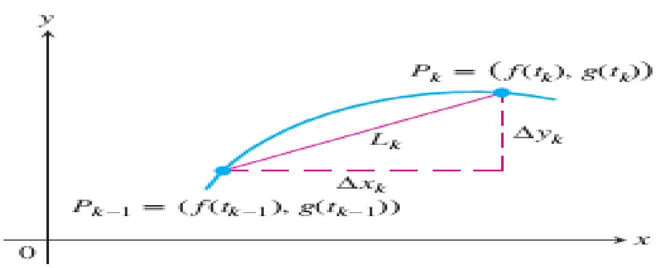


FIGURE 6.25 The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

DEFINITION Length of a Parametric Curve
If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

EXAMPLE 1

Find the length of the circle of radius r defined parametrically by

$$x = r\cos t \qquad \text{and} \qquad y = r\sin t, \qquad 0 \le t \le 2\pi.$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$$\frac{dx}{dt} = -r\sin t, \quad \frac{dy}{dt} = r\cos t$$

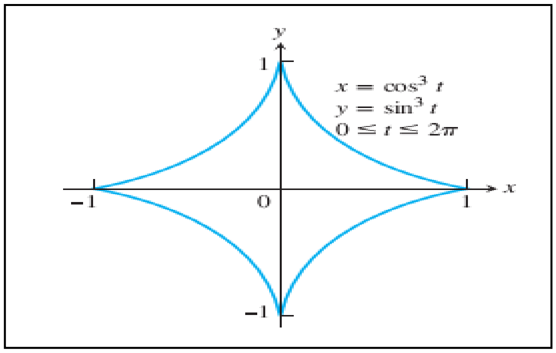
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2.$$

$$L = \int_0^{2\pi} \sqrt{r^2}\, dt = r\Big[t\Big]_0^{2\pi} = 2\pi r.$$

EXAMPLE 2

Find the length for the figure

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$



Solution

- ❖ Because of the curve's symmetry with respect to the coordinate axes,
- ❖ The length is four times the length of the first-quadrant portion.

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t(-\sin t)]^2 = 9\cos^4 t\sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t(\cos t)]^2 = 9\sin^4 t\cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^2 t\sin^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}}$$

$$= \sqrt{9\cos^2 t\sin^2 t}$$

$$= 3|\cos t\sin t|$$

$$= 3\cos t\sin t.$$

$\cos t \sin t \ge 0$ for
$0 \le t \le \pi/2$

Therefore,

$$\text{Length of first-quadrant portion} = \int_0^{\pi/2} 3\cos t \sin t \, dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t \, dt \qquad \cos t \sin t = (1/2)\sin 2t$$

$$= -\frac{3}{4}\cos 2t \Big]_0^{\pi/2} = \frac{3}{2}.$$

The length of the astroid is four times this: $4(3/2) = 6$. ■

Length of a Curve y = f(x)

Given a continuously differentiable function $y = f(x), a \le x \le b$,,,,, $x = t$

$$x = t \quad \text{and} \quad y = f(t), \quad a \le t \le b,$$

a special case of what we considered before. Then,

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = f'(t).$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = f'(t)$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 + [f'(t)]^2$$

$$= 1 + \left(\frac{dy}{dx}\right)^2$$

$$= 1 + [f'(x)]^2.$$

Formula for the Length of $y = f(x)$, $a \le x \le b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (2)$$

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution We use Equation (2) with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$

The length of the curve from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

Eq. (2) with $a = 0$, $b = 1$

$$= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6}.$$

Let $u = 1 + 8x$, integrate, and replace u by $1 + 8x$.

If Discontinuities in dy/dx

- At a point on a curve where dy / dx fails to exist.
- dx/ dy may exist and we may be able to find the curve's length by expressing x as a function of y.
- and applying the following analogue Equation :

Formula for the Length of $x = g(y)$, $c \le y \le d$

If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \qquad (3)$$

Example

Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (2).

We therefore rewrite the equation to express x in terms of y:

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2} \qquad \text{Raise both sides to the power 3/2.}$$

$$x = 2y^{3/2}. \qquad \text{Solve for } x.$$

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Figure 6.27).

The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

is continuous on [0, 1]. We may therefore use Equation (3) to find the curve's length:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 \sqrt{1 + 9y}\,dy \qquad \text{Eq. (3) with } c = 0,\ d = 1.$$

$$= \frac{1}{9}\cdot\frac{2}{3}(1 + 9y)^{3/2}\Big]_0^1 \qquad \text{Let } u = 1 + 9y,\ du/9 = dy, \text{ integrate, and substitute back.}$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.27.$$

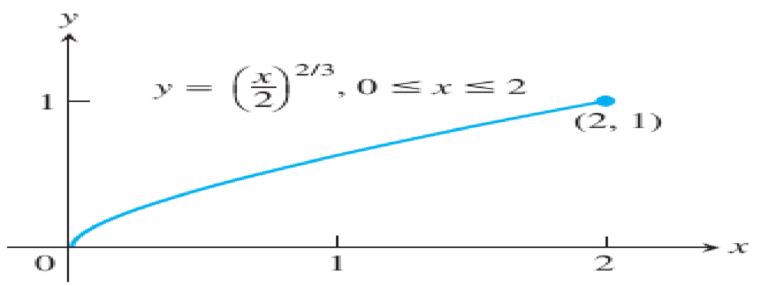


FIGURE 6.27 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$

Exer

Find the lengths of the curves in Exercises 1–6.

1. $x = 1 - t,\quad y = 2 + 3t,\quad -2/3 \le t \le 1$
2. $x = \cos t,\quad y = t + \sin t,\quad 0 \le t \le \pi$
3. $x = t^3,\quad y = 3t^2/2,\quad 0 \le t \le \sqrt{3}$
4. $x = t^2/2,\quad y = (2t + 1)^{3/2}/3,\quad 0 \le t \le 4$
5. $x = (2t + 3)^{3/2}/3,\quad y = t + t^2/2,\quad 0 \le t \le 3$
6. $x = 8\cos t + 8t\sin t,\quad y = 8\sin t - 8t\cos t,\quad 0 \le t \le \pi/2$

Finding Lengths of Curves

Find the lengths of the curves in Exercises 7–16. If you have a grapher, you may want to graph these curves to see what they look like.

7. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$
8. $y = x^{3/2}$ from $x = 0$ to $x = 4$
9. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)
10. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)
11. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)
12. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)
13. $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5,\quad 1 \le x \le 8$
14. $y = (x^3/3) + x^2 + x + 1/(4x + 4),\quad 0 \le x \le 2$
15. $x = \int_0^y \sqrt{\sec^4 t - 1}\,dt,\quad -\pi/4 \le y \le \pi/4$
16. $y = \int_{-2}^x \sqrt{3t^4 - 1}\,dt,\quad -2 \le x \le -1$

1. $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = 3 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-1)^2 + (3)^2} = \sqrt{10}$

$\Rightarrow \text{Length} = \int_{-2/3}^{1} \sqrt{10}\,dt = \sqrt{10}\,[t]_{-2/3}^{1} = \sqrt{10} - \left(-\frac{2}{3}\sqrt{10}\right) = \frac{5\sqrt{10}}{3}$

2. $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = 1 + \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (1 + \cos t)^2} = \sqrt{2 + 2\cos t}$

$\Rightarrow \text{Length} = \int_0^{\pi} \sqrt{2 + 2\cos t}\,dt = \sqrt{2}\int_0^{\pi} \sqrt{\left(\frac{1-\cos t}{1-\cos t}\right)(1 + \cos t)}\,dt = \sqrt{2}\int_0^{\pi} \sqrt{\frac{\sin^2 t}{1 - \cos t}}\,dt$

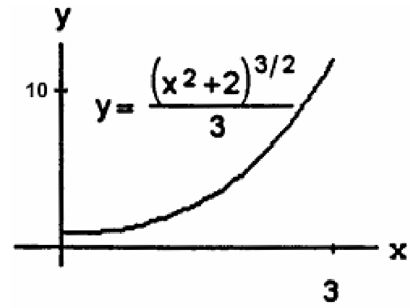
$= \sqrt{2}\int_0^{\pi} \frac{\sin t}{\sqrt{1-\cos t}}\,dt$ (since $\sin t \ge 0$ on $[0, \pi]$); $[u = 1 - \cos t \Rightarrow du = \sin t\,dt;\ t = 0 \Rightarrow u = 0,$
$t = \pi \Rightarrow u = 2] \rightarrow \sqrt{2}\int_0^2 u^{-1/2}\,du = \sqrt{2}\,\left[2u^{1/2}\right]_0^2 = 4$

7. $\frac{dy}{dx} = \frac{1}{3} \cdot \frac{3}{2}(x^2+2)^{1/2} \cdot 2x = \sqrt{(x^2+2)} \cdot x$

$$\Rightarrow L = \int_0^3 \sqrt{1+(x^2+2)x^2}\,dx = \int_0^3 \sqrt{1+2x^2+x^4}\,dx$$

$$= \int_0^3 \sqrt{(1+x^2)^2}\,dx = \int_0^3 (1+x^2)\,dx = \left[x + \frac{x^3}{3}\right]_0^3$$

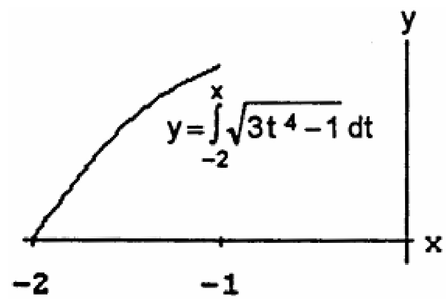
$$= 3 + \frac{27}{3} = 12$$



16. $\frac{dy}{dx} = \sqrt{3x^4-1} \Rightarrow \left(\frac{dy}{dx}\right)^2 = 3x^4 - 1$

$$\Rightarrow L = \int_{-2}^{-1} \sqrt{1+(3x^4-1)}\,dx = \int_{-2}^{-1} \sqrt{3}\,x^2\,dx$$

$$= \sqrt{3}\left[\frac{x^3}{3}\right]_{-2}^{-1} = \frac{\sqrt{3}}{3}\left[-1-(-2)^3\right] = \frac{\sqrt{3}}{3}(-1+8) = \frac{7\sqrt{3}}{3}$$



4- SURFACE AREA OF REVOLUTION

DEFINITION Surface Area for Revolution About the x-Axis

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx. \quad (3)$$

EXAMPLE 1 Applying the Surface Area Formula

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis (Figure 6.48).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \qquad b = 2, \qquad y = 2\sqrt{x}, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x + 1}{x}} = \frac{\sqrt{x + 1}}{\sqrt{x}}.$$

With these substitutions,

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\,\frac{\sqrt{x + 1}}{\sqrt{x}}\, dx = 4\pi \int_1^2 \sqrt{x + 1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x + 1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$

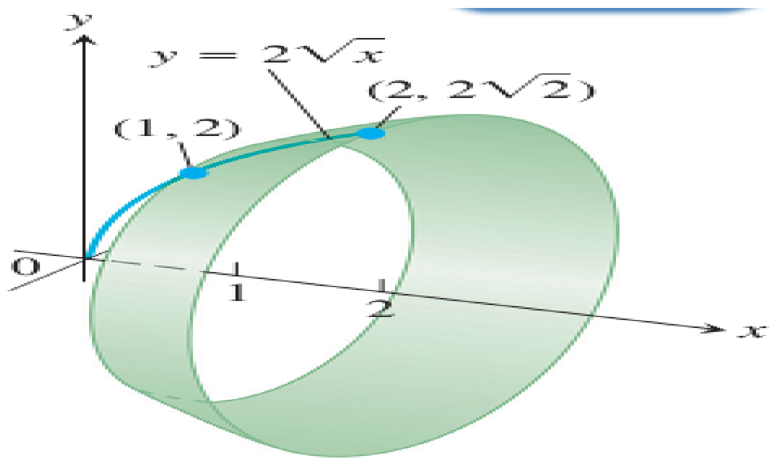


FIGURE 6.48 In Example 1 we calculate the area of this surface.

Surface Area for Revolution About the y-Axis
If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

EXAMPLE The line segment $x = 1 - y, 0 \leq y \leq 1$ is revolved about the y-axis to generate the cone in Figure 6.49. Find its lateral surface area.

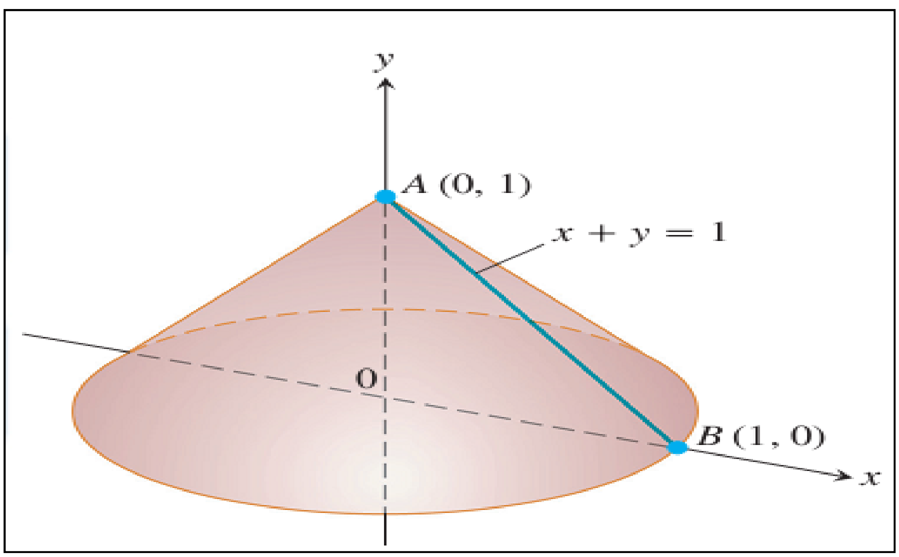
$$c = 0, \qquad d = 1, \qquad x = 1 - y, \qquad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi(1 - y)\sqrt{2}\, dy$$

$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right)$$

$$= \pi\sqrt{2}.$$



Exer

Finding Surface Areas

9. Find the lateral (side) surface area of the cone generated by revolving the line segment $y = x/2$, $0 \le x \le 4$, about the x-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

10. Find the lateral surface area of the cone generated by revolving the line segment $y = x/2$, $0 \le x \le 4$ about the y-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

11. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2)$, $1 \le x \le 3$, about the x-axis. Check your result with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

12. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2)$, $1 \le x \le 3$, about the y-axis. Check your result with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

Find the areas of the surfaces generated by revolving the curves in Exercises 13–22 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

13. $y = x^3/9$, $0 \le x \le 2$; x-axis

14. $y = \sqrt{x}$, $3/4 \le x \le 15/4$; x-axis

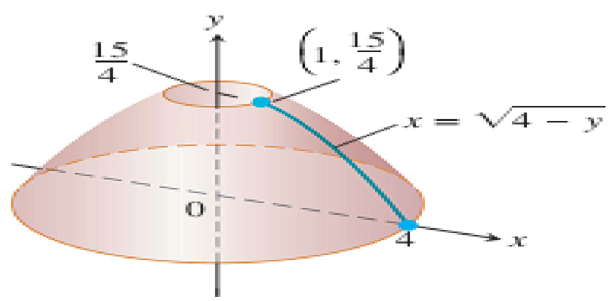
15. $y = \sqrt{2x - x^2}$, $0.5 \le x \le 1.5$; x-axis

16. $y = \sqrt{x+1}$, $1 \le x \le 5$; x-axis

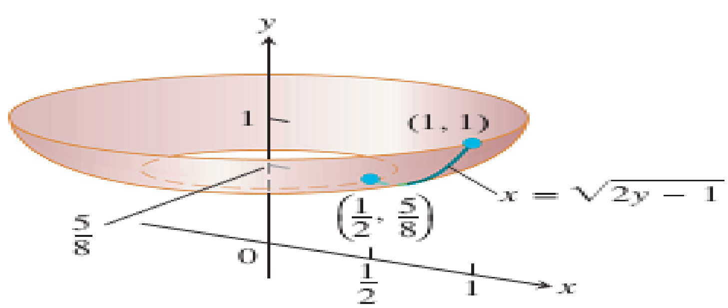
17. $x = y^3/3$, $0 \le y \le 1$; y-axis

18. $x = (1/3)y^{3/2} - y^{1/2}$, $1 \le y \le 3$; y-axis

19. $x = 2\sqrt{4 - y}$, $0 \le y \le 15/4$; y-axis



20. $x = \sqrt{2y - 1}$, $5/8 \le y \le 1$; y-axis



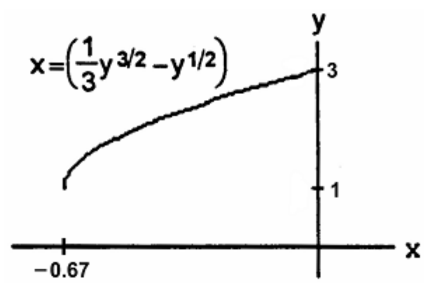
21. $x = (y^4/4) + 1/(8y^2)$, $1 \le y \le 2$; x-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dy, and evaluate the integral $S = \int 2\pi y\, ds$ with appropriate limits.)

22. $y = (1/3)(x^2 + 2)^{3/2}$, $0 \le x \le \sqrt{2}$; y-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dx, and evaluate the integral $S = \int 2\pi x\, ds$ with appropriate limits.)

9. $y = \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$; $S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \Rightarrow S = \int_0^4 2\pi \left(\frac{x}{2}\right) \sqrt{1 + \frac{1}{4}}\, dx = \frac{\pi\sqrt{5}}{2} \int_0^4 x\, dx$
$= \frac{\pi\sqrt{5}}{2} \left[\frac{x^2}{2}\right]_0^4 = 4\pi\sqrt{5}$; Geometry formula: base circumference $= 2\pi(2)$, slant height $= \sqrt{4^2 + 2^2} = 2\sqrt{5}$
$\Rightarrow$ Lateral surface area $= \frac{1}{2}(4\pi)\left(2\sqrt{5}\right) = 4\pi\sqrt{5}$ in agreement with the integral value

10. $y = \frac{x}{2} \Rightarrow x = 2y \Rightarrow \frac{dx}{dy} = 2$; $S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^2 2\pi \cdot 2y\sqrt{1 + 2^2}\, dy = 4\pi\sqrt{5} \int_0^2 y\, dy = 2\pi\sqrt{5}\,[y^2]_0^2$
$= 2\pi\sqrt{5} \cdot 4 = 8\pi\sqrt{5}$; Geometry formula: base circumference $= 2\pi(4)$, slant height $= \sqrt{4^2 + 2^2} = 2\sqrt{5}$
$\Rightarrow$ Lateral surface area $= \frac{1}{2}(8\pi)\left(2\sqrt{5}\right) = 8\pi\sqrt{5}$ in agreement with the integral value

18. $x = \left(\frac{1}{3} y^{3/2} - y^{1/2}\right) \le 0$, when $1 \le y \le 3$. To get positive area, we take $x = -\left(\frac{1}{3} y^{3/2} - y^{1/2}\right)$



$\Rightarrow \frac{dx}{dy} = -\frac{1}{2}\left(y^{1/2} - y^{-1/2}\right) \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + y^{-1}\right)$
$\Rightarrow S = -\int_1^3 2\pi \left(\frac{1}{3} y^{3/2} - y^{1/2}\right) \sqrt{1 + \frac{1}{4}\left(y - 2 + y^{-1}\right)}\, dy$
$= -2\pi \int_1^3 \left(\frac{1}{3} y^{3/2} - y^{1/2}\right) \sqrt{\frac{1}{4}\left(y + 2 + y^{-1}\right)}\, dy$
$= -2\pi \int_1^3 \left(\frac{1}{3} y^{3/2} - y^{1/2}\right) \frac{\sqrt{(y^{1/2} + y^{-1/2})^2}}{2}\, dy = -\pi \int_1^3 y^{1/2}\left(\frac{1}{3} y - 1\right)\left(y^{1/2} + \frac{1}{y^{1/2}}\right) dy = -\pi \int_1^3 \left(\frac{1}{3} y - 1\right)(y + 1)\, dy$
$= -\pi \int_1^3 \left(\frac{1}{3} y^2 - \frac{2}{3} y - 1\right) dy = -\pi \left[\frac{y^3}{9} - \frac{y^2}{3} - y\right]_1^3 = -\pi\left[\left(\frac{27}{9} - \frac{9}{3} - 3\right) - \left(\frac{1}{9} - \frac{1}{3} - 1\right)\right] = -\pi\left(-3 - \frac{1}{9} + \frac{1}{3} + 1\right)$
$= -\frac{\pi}{9}(-18 - 1 + 3) = \frac{16\pi}{9}$

19. $\frac{dx}{dy} = \frac{-1}{\sqrt{4-y}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4-y} \Rightarrow S = \int_0^{15/4} 2\pi \cdot 2\sqrt{4-y}\sqrt{1 + \frac{1}{4-y}}\, dy = 4\pi \int_0^{15/4} \sqrt{(4-y) + 1}\, dy$
$= 4\pi \int_0^{15/4} \sqrt{5-y}\, dy = -4\pi\left[\frac{2}{3}(5-y)^{3/2}\right]_0^{15/4} = -\frac{8\pi}{3}\left[\left(5 - \frac{15}{4}\right)^{3/2} - 5^{3/2}\right] = -\frac{8\pi}{3}\left[\left(\frac{5}{4}\right)^{3/2} - 5^{3/2}\right]$
$= \frac{8\pi}{3}\left(5\sqrt{5} - \frac{5\sqrt{5}}{8}\right) = \frac{8\pi}{3}\left(\frac{40\sqrt{5} - 5\sqrt{5}}{8}\right) = \frac{35\pi\sqrt{5}}{3}$

CHAPTER SIX

APPROXIMATIONS INTEGRAL OR (NUMEIRCAL INTEGRAL)

- When we cannot evaluate a definite integral with an anti derivative,
- We use numerical methods such as the Trapezoidal Rule and Simpson's Rule developed in this chapter.

1- TRAPEZOIDAL Rule

- When we cannot find a workable anti derivative for a function f that we have to integrate.
- We partition the interval of integration.
- Replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the integral of f.
- We therefore, assume that the length of each subinterval is

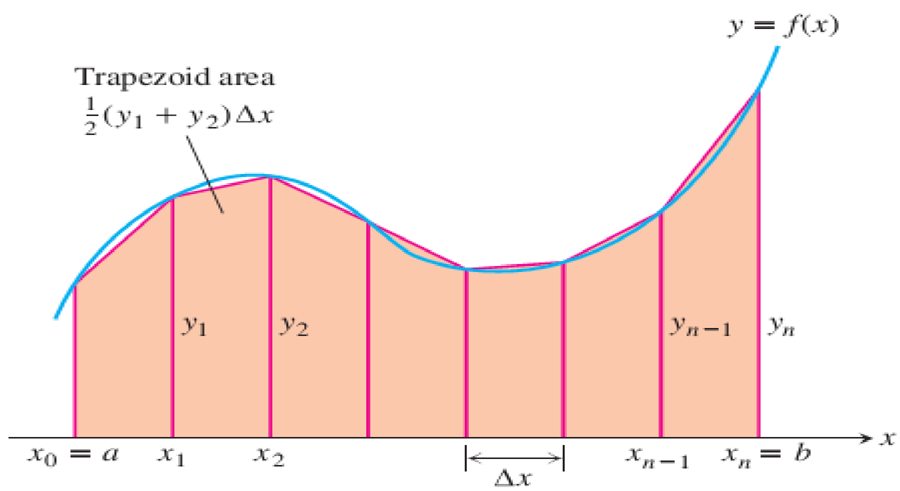
$$\Delta x = \frac{b-a}{n}.$$

or

$$N = \frac{(b-a)}{h}.$$

The length **(Δx)** is called the **step size or mesh size**. The area of the trapezoid that lies above the ***ith*** subinterval is

$$\Delta x\left(\frac{y_{i-1}+y_i}{2}\right) = \frac{\Delta x}{2}(y_{i-1}+y_i),$$



$$
\begin{aligned}
T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots \\
&\quad + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\
&= \Delta x \left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\
&= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n),
\end{aligned}
$$

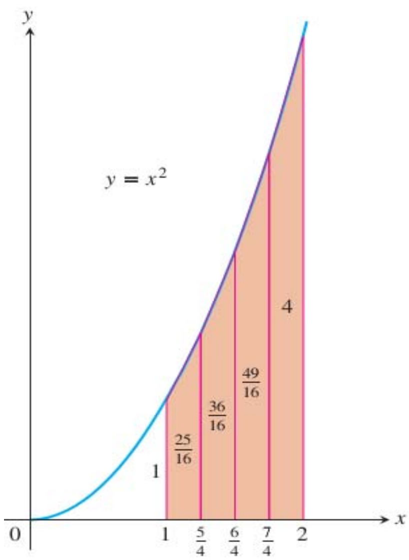
The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$
where $\Delta x = (b - a)/n$.



x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

EXAMPLE 1 Applying the Trapezoidal Rule

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

Solution Partition [1, 2] into four subintervals of equal length (Figure 8.11). Then evaluate $y = x^2$ at each partition point (Table 8.3).

Using these y values, $n = 4$, and $\Delta x = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$
\begin{aligned}
T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \\
&= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) \\
&= \frac{75}{32} = 2.34375.
\end{aligned}
$$

The exact value of the integral is

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\bigg]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

EXAMPLE 2 Averaging Temperatures

An observer measures the outside temperature every hour from noon until midnight, recording the temperatures in the following table.

Time	N	1	2	3	4	5	6	7	8	9	10	11	M
Temp	63	65	66	68	70	69	68	68	65	64	62	58	55

What was the average temperature for the 12-hour period?

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

$$\Delta x = \frac{b-a}{n}.$$

Δx = 12-0/12 = 1

$$\begin{aligned}T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{11} + y_{12}\right)\\ &= \frac{1}{2}\left(63 + 2\cdot 65 + 2\cdot 66 + \cdots + 2\cdot 58 + 55\right)\\ &= 782\end{aligned}$$

$$\operatorname{av}(f) \approx \frac{1}{b-a}\cdot T = \frac{1}{12}\cdot 782 \approx 65.17.$$

Example

$$\int_1^3 x^2\,\mathrm{d}x.$$

$h = (3 - 1)/4 = 0.5$

$x_0 = 1 \quad y_0 = (1)^2 = 1$
$x_1 = 1.5 \quad y_1 = (1.5)^2 = 2.25$
$x_2 = 2 \quad y_2 = (2)^2 = 4$
$x_3 = 2.5 \quad y_3 = (2.5)^2 = 6.25$
$x_4 = 3 \quad y_4 = (3)^2 = 9.$

Using the formula for the trapezoidal rule:

$$A = h(\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{N-1} + \tfrac{1}{2}y_N)$$

we get

$$A = 0.5(0.5 + 2.25 + 4 + 6.25 + 4.5) = 8.75.$$

Hence, by the trapezoidal rule:

$$\int_1^3 x^2 \, dx \approx 8.75.$$

Example 1:

Use the Trapezoidal Rule with $n = 6$ to estimate $\int_0^3 e^{x^2-4}\,dx$. Give answer with 3 decimals places.

$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

$$\int_0^3 e^{x^2-4}\,dx \approx \frac{h}{2}[\text{total}]$$

$$\approx \frac{0.5}{2}[169.9011083] \approx 42.475$$

x	$f(x) = e^{x^2-4}$
0	$f(0) = 0.01832$
0.5	$2f(0.5) = 2(0.023517745) = 0.047035491$
1	$2f(1) = 2(0.049787068) = 0.099574136$
1.5	$2f(1.5) = 2(0.173773943) = 0.347547886$
2	$2f(2) = 2(1) = 2$
2.5	$2f(2.5) = 2(9.487735836) = 18.97547167$
3	$f(3) = 148.4131591$
Total =	169.9011083

Example 2:

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_{-1}^3 \frac{1}{\ln(x+4)}\,dx$. Give answer with 3 decimals places.

$$h = \frac{b-a}{n} = \frac{3-(-1)}{4} = 1$$

$$\int_{-1}^3 \frac{1}{\ln(x+4)}\,dx \approx \frac{h}{2}[\text{total}]$$

$$\approx \frac{1}{2}[5.225723731] \approx 2.613$$

x	$f(x) = \dfrac{1}{\ln(x+4)}$
-1	$f(-1) = 0.910239226$
0	$2f(0) = 2(0.72134752) = 1.442695041$
1	$2f(1) = 2(0.621334934) = 1.242669869$
2	$2f(2) = 2(0.558110626) = 1.116221253$
3	$f(3) = 0.513898342$
Total =	5.225723731

2- Simpson's Rule

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b-a)/n$.

TABLE 8.4

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

EXAMPLE 5 Applying Simpson's Rule

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

Solution Partition [0, 2] into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.4). Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

$$\begin{aligned} S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right) \\ &= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right) \\ &= 32\frac{1}{12}. \end{aligned}$$

Example 4:

Use the Simpson's Rule with $n = 6$ to estimate $\int_0^3 e^{x^2-4}\,dx$. Give answer with 3 decimals places.

$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

$$\int_0^3 e^{x^2-4}\,dx \approx \frac{h}{3}[\text{total}]$$

$$\approx \frac{0.5}{3}[189.2711633] \approx 31.545$$

x	$f(x) = e^{x^2-4}$
0	$f(0) = 0.01832$
0.5	$4f(0.5) = 4(0.023517745) = 0.094070983$
1	$2f(1) = 2(0.049787068) = 0.099574136$
1.5	$4f(1.5) = 4(0.173773943) = 0.695095773$
2	$2f(2) = 2(1) = 2$
2.5	$4f(2.5) = 4(9.487735836) = 37.95094335$
3	$f(3) = 148.4131591$
Total =	189.2711633

Example 6:

Use the Simpson's Rule with $n = 4$ to estimate $\int_0^4 \frac{10}{\sqrt{x^3+8}}\,dx$. Give answer with 3 decimals places.

$$h=\frac{b-a}{n}=\frac{4-0}{4}=1$$

$$\int_0^4 \frac{10}{\sqrt{x^3+8}}\,dx \approx \frac{h}{3}[\text{total}]$$

$$\approx \frac{1}{3}[29.80861258] \approx 9.936$$

x	$f(x)=\frac{10}{\sqrt{x^3+8}}$
0	$f(0)=3.535533906$
1	$4f(1)=4(3.333333333)=13.33333333$
2	$2f(2)=2(2.5)=5$
3	$4f(3)=4(1.690308509)=6.761234038$
4	$f(4)=1.178511302$
Total $=$	29.80861258

In Exercises 11–14, use the tabulated values of the integrand to estimate the integral with (a) the Trapezoidal Rule and (b) Simpson's Rule with steps. Round your answers to five decimal places .

11. $\int_0^1 x\sqrt{1-x^2}\,dx$

x	$x\sqrt{1-x^2}$
0	0.0
0.125	0.12402
0.25	0.24206
0.375	0.34763
0.5	0.43301
0.625	0.48789
0.75	0.49608
0.875	0.42361
1.0	0

12. $\int_0^3 \frac{\theta}{\sqrt{16+\theta^2}}\,d\theta$

θ	$\theta/\sqrt{16+\theta^2}$
0	0.0
0.375	0.09334
0.75	0.18429
1.125	0.27075
1.5	0.35112
1.875	0.42443
2.25	0.49026
2.625	0.58466
3.0	0.6

13. $\int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2+\sin t)^2}\,dt$

t	$(3\cos t)/(2+\sin t)^2$
−1.57080	0.0
−1.17810	0.99138
−0.78540	1.26906
−0.39270	1.05961
0	0.75
0.39270	0.48821
0.78540	0.28946
1.17810	0.13429
1.57080	0

14. $\int_{\pi/4}^{\pi/2} (\csc^2 y)\sqrt{\cot y}\,dy$

y	$(\csc^2 y)\sqrt{\cot y}$
0.78540	2.0
0.88357	1.51606
0.98175	1.18237
1.07992	0.93998
1.17810	0.75402
1.27627	0.60145
1.37445	0.46364
1.47262	0.31688
1.57080	0

14. (a) $n = 8 \Rightarrow \Delta x = \frac{\pi}{32} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{64}$;

$\sum mf(y_i) = 1(2.0) + 2(1.51606) + 2(1.18237) + 2(0.93998) + 2(0.75402) + 2(0.60145) + 2(0.46364)$
$+ 2(0.31688) + 1(0) = 13.5488 \Rightarrow T \approx \frac{\pi}{64}(13.5488) = 0.66508$

(b) $n = 8 \Rightarrow \Delta x = \frac{\pi}{32} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{96}$;

$\sum mf(y_i) = 1(2.0) + 4(1.51606) + 2(1.18237) + 4(0.93988) + 2(0.75402) + 4(0.60145) + 2(0.46364)$
$+ 4(0.31688) + 1(0) = 20.29734 \Rightarrow S \approx \frac{\pi}{96}(20.29734) = 0.66423$

11. (a) $n = 8 \Rightarrow \Delta x = \frac{1}{8} \Rightarrow \frac{\Delta x}{2} = \frac{1}{16}$;

$\sum mf(x_i) = 1(0.0) + 2(0.12402) + 2(0.24206) + 2(0.34763) + 2(0.43301) + 2(0.48789) + 2(0.49608)$
$+ 2(0.42361) + 1(0) = 5.1086 \Rightarrow T = \frac{1}{16}(5.1086) = 0.31929$

(b) $n = 8 \Rightarrow \Delta x = \frac{1}{8} \Rightarrow \frac{\Delta x}{3} = \frac{1}{24}$;

$\sum mf(x_i) = 1(0.0) + 4(0.12402) + 2(0.24206) + 4(0.34763) + 2(0.43301) + 4(0.48789) + 2(0.49608)$
$+ 4(0.42361) + 1(0) = 7.8749 \Rightarrow S = \frac{1}{24}(7.8749) = 0.32812$

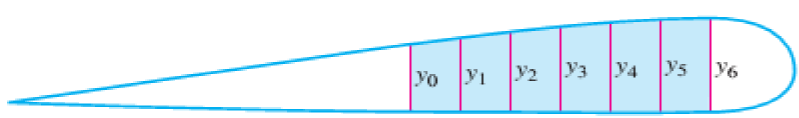
27. Volume of water in a swimming pool A rectangular swimming pool is 30 ft wide and 50 ft long. The table shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$, applied to the integral

$$V = \int_0^{50} 30 \cdot h(x)\,dx.$$

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

27. $\frac{5}{2}(6.0 + 2(8.2) + 2(9.1) \ldots + 2(12.7) + 13.0)(30) = 15{,}990 \text{ ft}^3$.

31. Wing design The design of a new airplane requires a gasoline tank of constant cross-sectional area in each wing. A scale drawing of a cross-section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft^3. Estimate the length of the tank.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

32. Oil consumption on Pathfinder Island A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced.

Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

Day	**Oil consumption rate (liters/h)**
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

50. The length of one arch of the curve $y = \sin x$ is given by

$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx.$$

Estimate L by Simpson's Rule with $n = 8$.

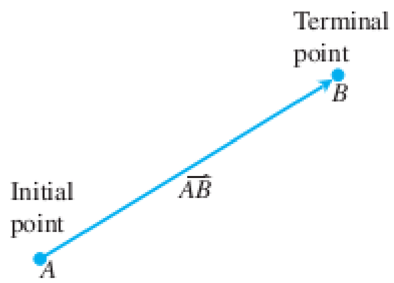
31. Using Simpson's Rule, $\Delta x = 1 \Rightarrow \frac{\Delta x}{3} = \frac{1}{3}$;
$\sum my_i = 33.6 \Rightarrow$ Cross Section Area $\approx \frac{1}{3}(33.6)$
$= 11.2\ \text{ft}^2$. Let x be the length of the tank. Then the Volume $V = (\text{Cross Sectional Area})\,x = 11.2x$.
Now 5000 lb of gasoline at $42\ \text{lb/ft}^3$
$\Rightarrow V = \frac{5000}{42} = 119.05\ \text{ft}^3$
$\Rightarrow 119.05 = 11.2x \Rightarrow x \approx 10.63$ ft

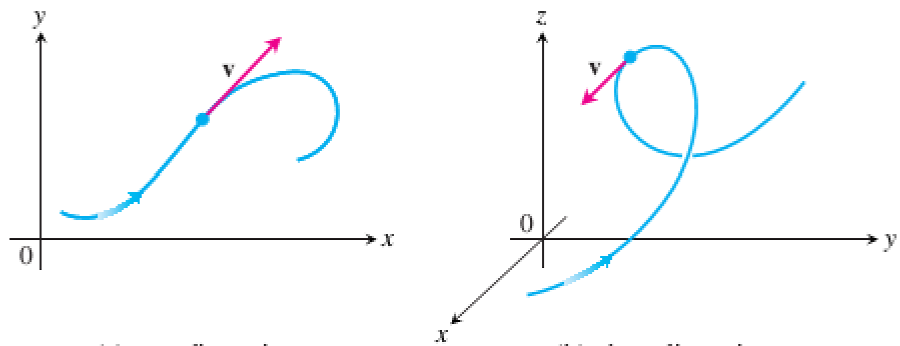
	x_i	y_i	m	my_i
x_0	0	1.5	1	1.5
x_1	1	1.6	4	6.4
x_2	2	1.8	2	3.6
x_3	3	1.9	4	7.6
x_4	4	2.0	2	4.0
x_5	5	2.1	4	8.4
x_6	6	2.1	1	2.1

CHAPTER 7

VECTOR ALGEBRA

Component Form





(a) two dimensions (b) three dimensions

DEFINITIONS Vector, Initial and Terminal Point, Length

A **vector** in the plane is a directed line segment. The directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B; its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

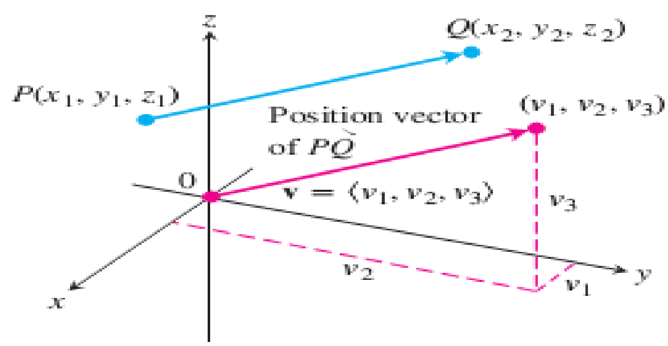
DEFINITION Component Form

If **v** is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the **component form** of **v** is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If **v** is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the **component form** of **v** is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$



$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ initial point is $P(x_1, y_1, z_1)$ terminal point is $Q(x_2, y_2, z_2)$

Then

V1= X2 - X1 ,,,V2 = Y2 - Y1 ,,,,, V3= Z2 - Z1 are the components of $\overrightarrow{PQ}$.

In summary, given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standard position vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ equal to $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2, -z_1 \rangle.$$

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3, 4, 1)$ an terminal point $Q(-5, 2, 2)$.

Solution

(a) The standard position vector $\mathbf{v}$ representing $\overrightarrow{PQ}$ has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2, \qquad v_2 = y_2 - y_1 = 2 - 4 = -2,$$

and

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form of $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle -2, -2, 1 \rangle.$$

(b) The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

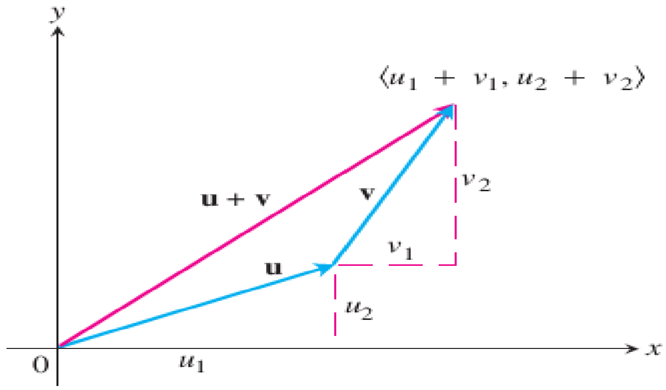
Vector Algebra Operations

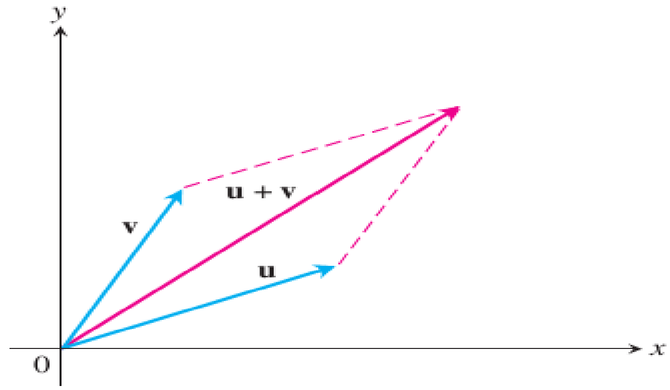
DEFINITIONS Vector Addition and Multiplication of a Vector by a Scalar

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

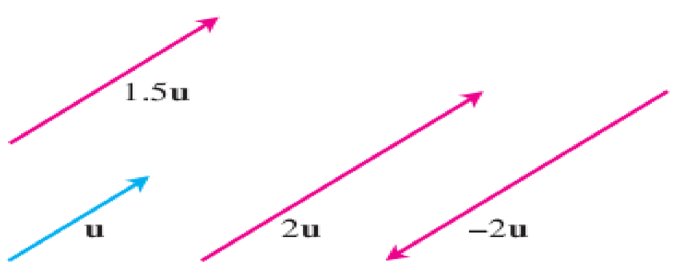
Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$



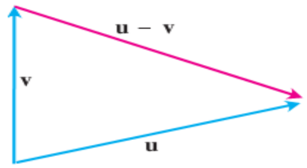


If $k > 0$, then $k\mathbf{u}$ has the same direction as $\mathbf{u}$; if $k < 0$, then the direction of $k\mathbf{u}$ is opposite to that of $\mathbf{u}$. Comparing the lengths of $\mathbf{u}$ and $k\mathbf{u}$, we see that

$$
\begin{aligned}
|k\mathbf{u}| &= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} = \sqrt{k^2(u_1{}^2 + u_2{}^2 + u_3{}^2)} \\
&= \sqrt{k^2}\sqrt{u_1{}^2 + u_2{}^2 + u_3{}^2} = |k|\,|\mathbf{u}|.
\end{aligned}
$$



The length of $k\mathbf{u}$ is the absolute value of the scalar k times the length of $\mathbf{u}$. The vector $(-1)\mathbf{u} = -\mathbf{u}$ has the same length as $\mathbf{u}$ but points in the opposite direction.



$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

Note that $(\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u}$, so adding the vector $(\mathbf{u} - \mathbf{v})$ to $\mathbf{v}$ gives $\mathbf{u}$

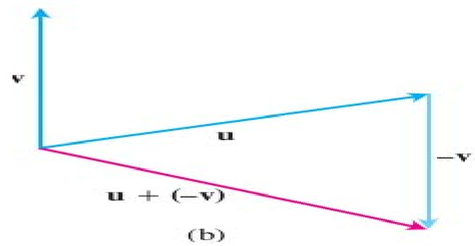


FIGURE 12.14 (a) The vector $\mathbf{u} - \mathbf{v}$, when added to $\mathbf{v}$, gives $\mathbf{u}$. (b) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

$\mathbf{u} - \mathbf{v}$ as the sum $\mathbf{u} + (-\mathbf{v})$.

EXAMPLE 3 Performing Operations on Vectors

Let $\mathbf{u} = \langle -1, 3, 1\rangle$ and $\mathbf{v} = \langle 4, 7, 0\rangle$. Find

(a) $2\mathbf{u} + 3\mathbf{v}$ **(b)** $\mathbf{u} - \mathbf{v}$ **(c)** $\left|\frac{1}{2}\mathbf{u}\right|$.

Solution

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1\rangle + 3\langle 4, 7, 0\rangle = \langle -2, 6, 2\rangle + \langle 12, 21, 0\rangle = \langle 10, 27, 2\rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1\rangle - \langle 4, 7, 0\rangle = \langle -1 - 4, 3 - 7, 1 - 0\rangle = \langle -5, -4, 1\rangle$

(c) $\left|\frac{1}{2}\mathbf{u}\right| = \left|\left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right\rangle\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit Vectors

A vector $\mathbf{v}$ of length 1 is called a **unit vector**. The **standard unit vectors** are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \qquad \mathbf{j} = \langle 0, 1, 0 \rangle, \qquad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as follows:

$$\begin{aligned} \mathbf{v} = \langle v_1, v_2, v_3 \rangle &= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}. \end{aligned}$$

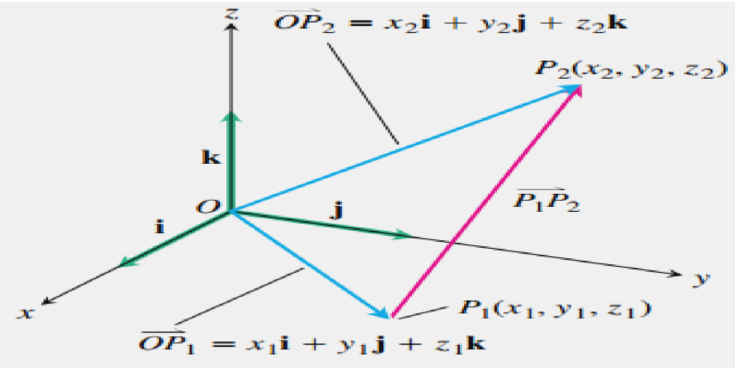
We call the scalar (or number) v_1 the **i-component** of the vector $\mathbf{v}$, v_2 the **j-component**, and v_3 the **k-component**. In component form, the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

Whenever $\mathbf{v} \neq \mathbf{0}$, its length $|\mathbf{v}|$ is not zero and

$$\left|\frac{1}{|\mathbf{v}|}\mathbf{v}\right| = \frac{1}{|\mathbf{v}|}|\mathbf{v}| = 1.$$

That is, $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of $\mathbf{v}$, called **the direction** of the nonzero vector $\mathbf{v}$.



The vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

Example

Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

The unit vector $\mathbf{u}$ is the direction of $\overrightarrow{P_1P_2}$.

EXAMPLE 6 Expressing Velocity as Speed Times Direction

If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times a unit vector in the direction of motion.

Solution Speed is the magnitude (length) of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ has the same direction as $\mathbf{v}$:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = \underbrace{5}_{\substack{\text{Length}\\ \text{(speed)}}}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{Direction of motion}}.$$

In summary, we can express any nonzero vector $\mathbf{v}$ in terms of its two important features, length and direction, by writing $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$.

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of $\mathbf{v}$;
2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ in terms of its length and direction.

EXAMPLE 7 A Force Vector

A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force $\mathbf{F}$ as a product of its magnitude and direction.

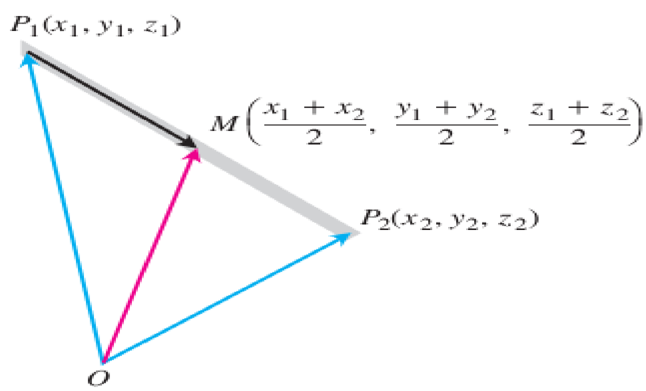
Solution The force vector has magnitude 6 and direction $\frac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\mathbf{F} = 6\frac{\mathbf{v}}{|\mathbf{v}|} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3}$$

$$= 6\left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\right).$$

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$



EXAMPLE 8 Finding Midpoints

The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2}\right) = (5, 1, 2).$$

EXERCISES 12.2

In Exercises 1–8, let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Find the **(a)** component form and **(b)** magnitude (length) of the vector.

1. $3\mathbf{u}$

2. $-2\mathbf{v}$

3. $\mathbf{u} + \mathbf{v}$

4. $\mathbf{u} - \mathbf{v}$

5. $2\mathbf{u} - 3\mathbf{v}$

6. $-2\mathbf{u} + 5\mathbf{v}$

7. $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v}$

8. $-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v}$

1. (a) $\langle 3(3), 3(-2) \rangle = \langle 9, -6 \rangle$
   (b) $\sqrt{9^2 + (-6)^2} = \sqrt{117} = 3\sqrt{13}$

2. (a) $\langle -2(-2), -2(5) \rangle = \langle 4, -10 \rangle$
   (b) $\sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29}$

In Exercises 9–16, find the component form of the vector.

9. The vector $\overrightarrow{PQ}$, where $P = (1, 3)$ and $Q = (2, -1)$

10. The vector $\overrightarrow{OP}$ where O is the origin and P is the midpoint of segment RS, where $R = (2, -1)$ and $S = (-4, 3)$

11. The vector from the point $A = (2, 3)$ to the origin

12. The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, where $A = (1, -1)$, $B = (2, 0)$, $C = (-1, 3)$, and $D = (-2, 2)$

9. $\langle 2 - 1, -1 - 3 \rangle = \langle 1, -4 \rangle$

10. $\left\langle \frac{2+(-4)}{2} - 0, \frac{-1+3}{2} - 0 \right\rangle = \langle -1, 1 \rangle$

12. $\overrightarrow{AB} = \langle 2 - 1, 0 - (-1) \rangle = \langle 1, 1 \rangle$, $\overrightarrow{CD} = \langle -2 - (-1), 2 - 3 \rangle = \langle -1, -1 \rangle$, $\overrightarrow{AB} + \overrightarrow{CD} = \langle 0, 0 \rangle$

13. The unit vector that makes an angle $\theta = 2\pi/3$ with the positive x-axis

14. The unit vector that makes an angle $\theta = -3\pi/4$ with the positive x-axis

15. The unit vector obtained by rotating the vector $\langle 0, 1 \rangle$ $120°$ counterclockwise about the origin

16. The unit vector obtained by rotating the vector $\langle 1, 0 \rangle$ $135°$ counterclockwise about the origin

13. $\left\langle \cos \frac{2\pi}{3}, \sin \frac{2\pi}{3} \right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$

14. $\left\langle \cos\left(-\frac{3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right) \right\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

15. This is the unit vector which makes an angle of $120° + 90° = 210°$ with the positive x-axis;
$\langle \cos 210°, \sin 210° \rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$

In Exercises 17–22, express each vector in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

17. $\overrightarrow{P_1P_2}$ if P_1 is the point $(5, 7, -1)$ and P_2 is the point $(2, 9, -2)$

18. $\overrightarrow{P_1P_2}$ if P_1 is the point $(1, 2, 0)$ and P_2 is the point $(-3, 0, 5)$

19. $\overrightarrow{AB}$ if A is the point $(-7, -8, 1)$ and B is the point $(-10, 8, 1)$

20. $\overrightarrow{AB}$ if A is the point $(1, 0, 3)$ and B is the point $(-1, 4, 5)$

21. $5\mathbf{u} - \mathbf{v}$ if $\mathbf{u} = \langle 1, 1, -1 \rangle$ and $\mathbf{v} = \langle 2, 0, 3 \rangle$

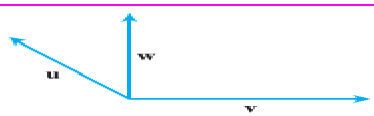
22. $-2\mathbf{u} + 3\mathbf{v}$ if $\mathbf{u} = \langle -1, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, 1, 1 \rangle$

17. $\overrightarrow{P_1P_2} = (2-5)\mathbf{i} + (9-7)\mathbf{j} + (-2-(-1))\mathbf{k} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

22. $-2\mathbf{u} + 3\mathbf{v} = -2\langle -1, 0, 2 \rangle + 3\langle 1, 1, 1 \rangle = \langle 2, 0, -4 \rangle + \langle 3, 3, 3 \rangle = \langle 5, 3, -1 \rangle = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

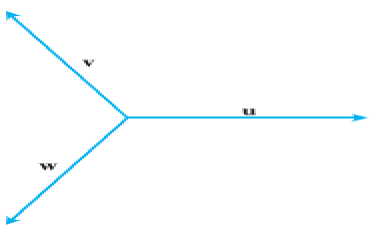
In Exercises 23 and 24, copy vectors **u**, **v**, and **w** head to tail as needed to sketch the indicated vector.

23.



a. $\mathbf{u} + \mathbf{v}$ b. $\mathbf{u} + \mathbf{v} + \mathbf{w}$
c. $\mathbf{u} - \mathbf{v}$ d. $\mathbf{u} - \mathbf{w}$

24.



a. $\mathbf{u} - \mathbf{v}$ b. $\mathbf{u} - \mathbf{v} + \mathbf{w}$
c. $2\mathbf{u} - \mathbf{v}$ d. $\mathbf{u} + \mathbf{v} + \mathbf{w}$

23)

In Exercises 25–30, express each vector as a product of its length and direction.

25. $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ **26.** $9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$

27. $5\mathbf{k}$ **28.** $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$

29. $\frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$ **30.** $\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}$

25. length $= |2\mathbf{i} + \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$, the direction is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \Rightarrow 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$

30. length $= \left|\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{3}}\right)^2} = 1$, the direction is $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$

$$\Rightarrow \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} = 1\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)$$

33. Find a vector of magnitude 7 in the direction of $\mathbf{v} = 12\mathbf{i} - 5\mathbf{k}$.

34. Find a vector of magnitude 3 in the direction opposite to the direction of $\mathbf{v} = (1/2)\mathbf{i} - (1/2)\mathbf{j} - (1/2)\mathbf{k}$.

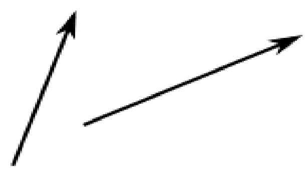
33. $|\mathbf{v}| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{13}\mathbf{v} = \frac{1}{13}(12\mathbf{i} - 5\mathbf{k}) \Rightarrow$ the desired vector is $\frac{7}{13}(12\mathbf{i} - 5\mathbf{k})$

34. $|\mathbf{v}| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{3}}{2}$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow$ the desired vector is $-3\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right)$
$= -\sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + \sqrt{3}\mathbf{k}$

Dot Product

- **The Dot Product gives a scalar (ordinary number) answer, and is sometimes called the scalar product.**

These are vectors :



They can be **multiplied** using the "**Dot Product**"

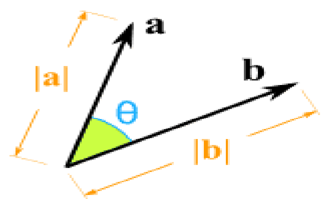
Calculating

The Dot Product is written using a central dot:

$$\mathbf{a} \cdot \mathbf{b}$$

This means the Dot Product of **a** and **b**

We can calculate the Dot Product of two vectors this way:

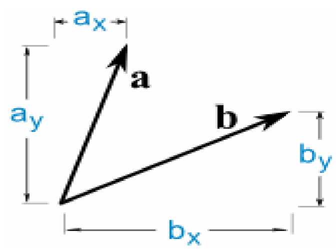


$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$$

Where:
$|\mathbf{a}|$ is the magnitude (length) of vector **a**
$|\mathbf{b}|$ is the magnitude (length) of vector **b**
θ is the angle between **a** and **b**

So we multiply the length of **a** times the length of **b**, then multiply by the cosine of the angle between **a** and **b**

OR we can calculate it this way:

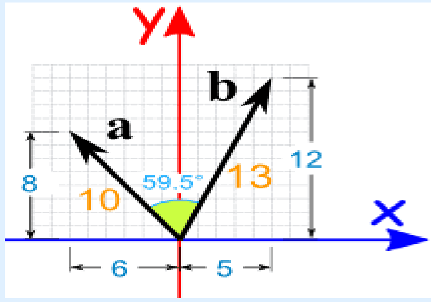


$$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$$

So we multiply the x's, multiply the y's, then add.

Both methods work!

Example: Calculate the dot product of vectors **a** and **b**:



$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$

$\mathbf{a} \cdot \mathbf{b} = 10 \times 13 \times \cos(59.5°)$

$\mathbf{a} \cdot \mathbf{b} = 10 \times 13 \times 0.5075...$

$\mathbf{a} \cdot \mathbf{b} = 65.98... = 66$ (rounded)

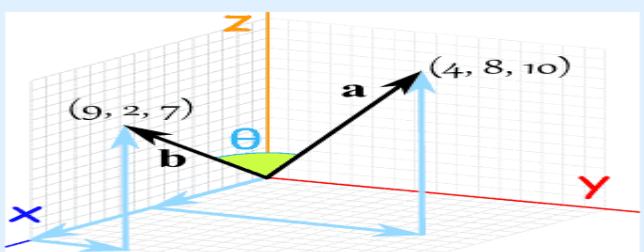
$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$

$\mathbf{a} \cdot \mathbf{b} = -6 \times 5 + 8 \times 12$

$\mathbf{a} \cdot \mathbf{b} = -30 + 96$

$\mathbf{a} \cdot \mathbf{b} = 66$

Example: Sam has measured the end-points of two poles, and wants to know **the angle between them**:



We have 3 dimensions, so don't forget the z-components:

$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y + a_z \times b_z$

$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y + a_z \times b_z$

$\rightarrow \mathbf{a} \cdot \mathbf{b} = 9 \times 4 + 2 \times 8 + 7 \times 10$

$\rightarrow \mathbf{a} \cdot \mathbf{b} = 36 + 16 + 70$

$\rightarrow \mathbf{a} \cdot \mathbf{b} = 122$

Now for the other formula:

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$

But what is $|\mathbf{a}|$? It is the magnitude, or length, of the vector **a**. We can use Pythagoras :

- $|\mathbf{a}| = \sqrt{(4^2 + 8^2 + 10^2)}$
- $|\mathbf{a}| = \sqrt{(16 + 64 + 100)}$
- $|\mathbf{a}| = \sqrt{180}$

Likewise for $|\mathbf{b}|$:

- $|\mathbf{b}| = \sqrt{(9^2 + 2^2 + 7^2)}$
- $|\mathbf{b}| = \sqrt{(81 + 4 + 49)}$
- $|\mathbf{b}| = \sqrt{134}$

And we know from the calculation above that $\mathbf{a} \cdot \mathbf{b} = 122$, so:

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$

$\rightarrow 122 = \sqrt{180} \times \sqrt{134} \times \cos(\theta)$

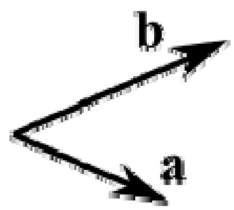
$\rightarrow \cos(\theta) = 122 / (\sqrt{180} \times \sqrt{134})$

$\rightarrow \cos(\theta) = 0.7855...$

$\rightarrow \theta = \cos^{-1}(0.7855...) = 38.2...°$

Cross Product

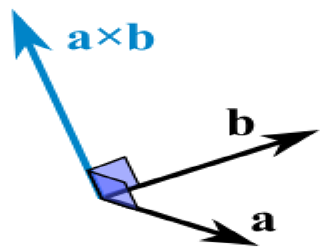
These are two vectors:



They can be **multiplied** using the "**Cross Product**"

☒ The Cross Product gives a vector answer, and is sometimes called the vector product.

The Cross Product **a × b** of two vectors is another vector that is at right angles to both:



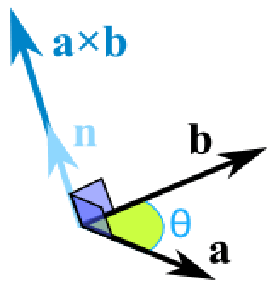
And it all happens in 3 dimensions!

Calculating

WE CAN CALCULATE THE CROSS PRODUCT THIS WAY:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$

- $|\mathbf{a}|$ is the magnitude (length) of vector **a**
- $|\mathbf{b}|$ is the magnitude (length) of vector **b**
- θ is the angle between **a** and **b**
- **n** is the unit vector at right angles to both **a** and **b**



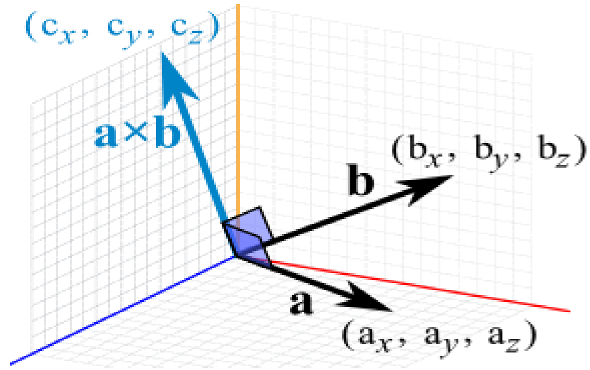
So the **length** is: the length of **a** times the length of **b** times the sine of the angle between **a** and **b**,

Then we multiply by the vector **n** to make sure it heads in the right **direction** (at right angles to both **a** and **b**).

OR WE CAN CALCULATE IT THIS WAY:

When **a** and **b** start at the origin point (0,0,0), the Cross Product will end at:

- $c_x = a_y b_z - a_z b_y$
- $c_y = a_z b_x - a_x b_z$
- $c_z = a_x b_y - a_y b_x$



Example: The cross product of a = (2,3,4) and b = (5,6,7)

- $c_x = a_y b_z - a_z b_y = 3\times7 - 4\times6 = -3$
- $c_y = a_z b_x - a_x b_z = 4\times5 - 2\times7 = 6$
- $c_z = a_x b_y - a_y b_x = 2\times6 - 3\times5 = -3$

Answer: **a × b** = (−3,6,−3)

EXAMPLE 1 Calculating Cross Products with Determinants

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

See chapter 12 in calculus book